IDENTIFICATION AND ESTIMATION OF A LARGE FACTOR MODEL WITH STRUCTURAL INSTABILITY

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Abstract

This paper tackles the identification and estimation of a large factor model with a single large break. We allow for a change in the number of factors which can be emerging or disappearing factors. We also allow for partial change in the factor loadings and for cases where a change in the factor loadings do not lead to extra pseudo factors. First, we propose a least squares estimator of the change point without requiring prior knowledge of the number of factors and observability of the factors. Rate of convergence and limit distribution of the proposed estimator are established under fairly general assumptions. We allow for cross-sectional and temporal dependence as well as heteroskedasticity of the idiosyncratic errors. Based on the estimated change point, we then split the sample into two subsamples and use each subsample to estimate the pre-break and post-break number of factors as well as the factors themselves. Consistency of the estimated number of factors and the estimated factor space are established under fairly general assumptions. The convergence rate of the estimated factor space is the same as in Bai and Ng (2002) obtained for a stable model. The finite sample performance of our estimators are investigated using Monte Carlo experiments.

Keywords: large factor model, structural instability, change point, rate of convergence, number of factors, model selection, factor space, panel data

JEL Classification: C13; C33.

1 INTRODUCTION

Large factor models where a large number of time series are simultaneously driven by a small number of unobserved factors, provide a powerful framework to analyze high dimensional data. In the past fifteen years, large factor models have been successfully used in business cycle analysis, consumer behavior analysis, asset pricing and economic monitoring and forecasting, see for example Bernanke, Boivin and Eliasz (2005), Lewbel (1991), Ross (1976) and Stock and Watson (2002b), to mention a few. Estimation theory of large factor models also experienced some breakthroughs, see Bai and Ng (2002) and Bai (2003), to mention a few. While most applications implicitly assume that the number of factors and factor loadings are stable, there is broad evidence of structural instability in macroeconomic and financial time series. Stock and Watson (2002a, 2009) argue that given the number of factors, standard principal component estimation of factors is still consistent if the magnitude of the factor loading break is small enough. Bates, Plagborg-Møller, Stock and Watson (2013) further argue that a sufficient condition for consistent estimation of the factor space is that the magnitude of the factor loading break should converge to zero asymptotically. The condition becomes increasingly stringent if one is to ensure the same convergence rate of the estimated factor space derived in Bai and Ng (2002). This plays a crucial role in subsequent forecasting and factor augmented regression models, and in ensuring consistent estimation of the number of factors. However, in many empirical applications, the magnitude of factor loading break could be large and the number of factors may also change over time. Examples include important economic events such as the European debt crisis, or political events such as the end of the cold war, or gradual but fundamental changes in economic structure due to technological progress, or policy change such as the end of China's one-child policy, to mention a few.

In the presence of **a** large factor loading break, estimation ignoring this instability leads to serious consequences. First, the estimated number of factors, using any existing method, e.g., Bai and Ng (2002), Onatski (2009, 2010) and Ahn and Horenstein (2013), is no longer consistent and tends to overestimate. This is because a factor model with unstable factor loadings can be represented by an equivalent model with extra pseudo factors but stable factor loadings. Moreover, the inconsistency of the estimated number of factors will be transmitted to the estimated factors. In such cases, it is hard to interpret the estimated factors, and forecasting performance deteriorate significantly. Consequently, a series of tests are proposed to test large factor loading break, including Breitung and Eickmeier (2011), Chen, Dolado and Gonzalo (2014), Han and Inoue (2014) and

Corradi and Swanson (2014). Once a large factor loading break has been detected, one still have to estimate the change point, determine the number of pre and post-break factors and estimate the factors themselves.

In fact, identification and estimation of factor model in the presence of structural instability have inherent difficulties. First, without knowing the change point, it is infeasible to consistently estimate the factors and factor loadings even if the number of pre-break and post-break factors were known. Second, existing change point estimation methods require knowledge of the number of regressors and observability of the regressors, see for example Bai (1994, 1997, 2010). Hence, to estimate the change point along this path, even if the number of pre-break and post-break factors were known, we still need at least a consistent estimator of the factors, which is infeasible without knowing the change point. Similarly, it is also infeasible to estimate the change point directly based on the estimated factor loadings since consistent estimation of factor loadings is infeasible without knowing the change point. For example, suppose that the number of factors is known and constant over time. In addition, suppose that after a certain time period, the factor loadings are all doubled. This model is observationally equivalent to the model where the factor loadings are constant over time, while the factors are all twice their true values after that time period. In this case, estimating the change point following Bai (1994, 1997) directly is not promising. Cheng, Liao and Schorfheide (2014) propose a shrinkage procedure that consistently estimates the number of pre and post-break factors and consistently detects factor loading breaks when the number of factors is constant, without requiring knowledge of the change point. This result, although is a significant breakthrough, does not generate a consistent estimate of the change point, nor consistent estimates of the factors or factor loadings when the change point is unknown. The main message from Cheng et al. (2014) is that we may behave as if the number of pre and post-break factors were known in constructing consistent estimators of the factors, the factor loadings and the change point. However, as we just discussed, the problem is still there even if the number of pre and post-break factors were known. In addition, Chen (2015) proposes a least square estimator that consistently estimates the break fraction.

In this paper we propose a least squares estimator of the change point without requiring knowledge of the number of factors and observability of the factors. The key observation behind the proposed estimator is that the change point of the factor loadings in the original model is the same as the change point of the second moment matrix of the factors in the equivalent model. Estimating the former can therefore be converted to estimating the latter, thereby circumventing the estimation of the original model. More specifically, our estimation procedure starts by estimating the number of factors and the factors themselves ignoring structural change. This leads us to identify the equivalent model. Based on the estimated factors, we then estimate the pre and post-break second moment matrix of the factors for all possible sample splits. The change point is estimated by minimizing the sum of squared residuals of this second moment matrix estimation among all possible sample splits. Rate of convergence and limit distribution of the proposed change point estimator are established under fairly general assumptions. These results are useful for analyzing and evaluating the effect of a policy change, for uncovering the underlying factors that lead to structural change, and for determining whether the response of economic variables are immediate or gradual, to mention a few applications. More importantly, these results illuminate identification and estimation of the whole model. Based on the estimated change point, we then split the sample into two subsamples and use each subsample to estimate the number of pre and post-break factors as well as the factor space. Although the estimated change point is inconsistent ($O_p(1)$), we are still able to establish the consistency of the estimated number of pre and post-break factors and the estimated factor space. Also, the convergence rate of the estimated factor space will be the same as the one in Bai and Ng (2002) for the stable model. This is because increasing the time series length (T), renders the estimation of the other parameters in the system more robust to misspecification of the change point. These results provide the foundation for subsequent analysis. Our estimation procedure also provides a systematic way of dealing with structural instability in factor based models, and will illuminate identification of dynamic factor models with structural instability. In addition, we also found that consistent estimation of the number of factors ignoring structural change can tolerate factor loading break of larger magnitude than that proved in Bates et al. (2013), if we accordingly adjust the magnitude of the penalty term of the information criteria in Bai and Ng (2002). This result is mainly of theoretical interest since in reality the relative magnitude of factor loading break and the factors themselves are hard to characterize.

Our assumptions are quite general, hence our results are widely applicable. We allow for cases with a change in the number of factors, which can be disappearing or emerging factors. We also allow for cases with only partial change in the factor loadings and cases in which a change in the factor loadings does not lead to extra pseudo factors. Our Assumptions 1-7 are either from or slight modification of Assumptions A-G in Bai (2003). These allow for cross-sectional and temporal dependence as well as heteroscedasticity of the idiosyncratic errors. The main extra assumption we impose is that the Hajek-Renyi inequality is applicable to the second moment process of the factors. As discussed in the next section, this assumption is more general than explicitly assuming a specific factor process and can be easily satisfied. It is also worth noting that for regularly behaved error term, our results do not rely on the relative speed of the number of subjects (N) and the time series length (T).

The rest of the paper is organized as follows. Section 2 introduces the model setup, notation and preliminaries. Section 3 discusses the equivalent representation and assumptions. Section 4 considers estimation of the change point. Section 5 considers estimation of the number of pre and post-break factors. Section 6 considers estimation of the factor space. Section 7 reports simulation results, while Section 8 concludes. All the proofs are given in the Appendix.

2 NOTATION AND PRELIMINARIES

Consider the following large factor model with structural change in the factor loadings:

$$x_{it} = \begin{cases} f_{0,t}'\lambda_{0,i} + f_{1,t}'\lambda_{1,i} + e_{i,t}, & if\ 1 \le t \le [\tau_0 T] \\ f_{0,t}'\lambda_{0,i} + f_{1,t}'\lambda_{2,i} + e_{i,t}, & if\ [\tau_0 T]+1 \le t \le T \end{cases} \quad \text{for } i = 1, ..., N \text{ and } t = 1, ..., T, \tag{1}$$

where $f_t = (f_{0,t}', f_{1,t}')'$. $f_{1,t}$ and $f_{0,t}$ are q and $r-q$ dimensional vectors of factors with and without structural change in their factor loadings, respectively. $\lambda_{0,i}$ is the factor loadings of subject i corresponding to $f_{0,t}$, $\lambda_{1,i}$ and $\lambda_{2,i}$ are factor loadings of subject i corresponding to $f_{1,t}$ before and after the structural change, respectively. It is easy to see that $r-q=0$ and $r-q>0$ correspond to the pure change case and the partial change case respectively. $e_{i,t}$ is the error term allowed to have temporal and cross-sectional dependence as well as heteroskedasticity. $\tau_0 \in (0,1)$ is the change fraction and $k_0 = [\tau_0 T]$ is the change point.

In matrix form, the model can be represented as:

$$X = \begin{bmatrix} F_1^0\Lambda_0' + F_1^1\Lambda_1' \\ F_2^0\Lambda_0' + F_2^1\Lambda_2' \end{bmatrix} + E, \tag{2}$$

where $F_1^0 = [f_{0,1}, ..., f_{0,[\tau_0 T]}]'$, $F_2^0 = [f_{0,[\tau_0 T]+1}, ..., f_{0,T}]'$, $F_1^1 = [f_{1,1}, ..., f_{1,[\tau_0 T]}]'$ and $F_2^1 = [f_{1,[\tau_0 T]+1}, ..., f_{1,T}]'$ are of dimensions $[\tau_0 T] \times (r-q)$, $[(1-\tau_0)T] \times (r-q)$, $[\tau_0 T] \times q$ and $[(1-\tau_0)T] \times q$, respectively. $\Lambda_0 = [\lambda_{0,1}, ..., \lambda_{0,N}]'$, $\Lambda_1 = [\lambda_{1,1}, ..., \lambda_{1,N}]'$ and $\Lambda_2 = [\lambda_{2,1}, ..., \lambda_{2,N}]'$ are of dimensions $N \times (r-q)$, $N \times q$ and $N \times q$, respectively, $E = [e_1, ..., e_T]'$ is of dimension $T \times N$. The matrices F_1^0, F_2^0, F_1^1, F_2^1, Λ_0, Λ_1, Λ_2 and E are all unknown. In addition, $\Lambda_{01} = [\Lambda_0, \Lambda_1] = (\lambda_{01,1}, ..., \lambda_{01,N})'$ and $\Lambda_{02} = [\Lambda_0, \Lambda_2] = (\lambda_{02,1}, ..., \lambda_{02,N})'$ are of dimension $N \times r$. Note that in general not only the factor loadings but also the number of factors may have structural change. In our representation, structural change in the number of factors is incorporated as a special case of structural change in factor loadings by allowing either Λ_{01} or Λ_{02} to be degenerate. In case the number of pre-break and post-break factors are r_1 and r_2 respectively, with $r = \max\{r_1, r_2\}$, f_t and λ_i are always r dimensional vectors and both Λ_{01} and Λ_{02} are of dimensions $N \times r$. If $r_1 < r_2$, some columns in Λ_{01} are zeros and the number of such columns is $r_2 - r_1$. In this case, Λ_{01} is degenerate and Λ_{02} is of full rank. Similarly, if $r_1 > r_2$, some columns in Λ_{02} are zeros and Λ_{01} is of full rank. If $r_1 = r_2$, both Λ_{01} and Λ_{02} are of full rank r. In addition, we want to point out that although cases with either disappearing factors or emerging factors are allowed for, cases with both disappearing factors and emerging factors are not necessarily identifiable within this mathematical setup. A model with s_1 disappearing factors and s_2 emerging factors can be observationally equivalent to a model with $s_1 - s_2$ disappearing factors.

Before moving forward, let us introduce the Hajek-Renyi inequality, which is a powerful and almost indispensible tool for calculating the stochastic order of sup-type terms. For a sequence of independent random variables $\{x_t, t = 1, ...\}$ with zero mean and finite variance, Hajek and Renyi (1955) proved that for any integers m and T,

$$P(\sup_{m \le k \le T} c_k \left| \sum\nolimits_{t=1}^{k} x_t \right| > M) \le \frac{1}{M^2}(c_m^2 \sum\nolimits_{t=1}^{m} \sigma_t^2 + \sum\nolimits_{t=m+1}^{T} c_t^2 \sigma_t^2), \tag{3}$$

where $\{c_k, k = 1, ...\}$ is a sequence of nondecreasing positive numbers and $Ex_t^2 = \sigma_t^2$. The Hajek-Renyi inequality was extended to various settings, including martingale difference, martingale, mixingale, linear process and vector-valued martingale, see Bai (1996). From expression (3), it is easy to see that if σ_t^2 is constant over time,

$$P(\sup_{m \le k \le T} \left| \frac{1}{k} \sum\nolimits_{t=1}^{k} x_t \right| > M) \le \frac{2\sigma^2}{M^2} \frac{1}{m},$$

hence when $m = 1$, $\sup_{1 \le k \le T} \left| \frac{1}{k} \sum_{t=1}^{k} x_t \right| = O_p(1)$ as $T \to \infty$ and when $m = [T\tau]$ for $\tau \in (0, 1)$, $\sup_{m \le k \le T} \left| \frac{1}{k} \sum_{t=1}^{k} x_t \right| = O_p(\frac{1}{\sqrt{T}})$ as $T \to \infty$; and

$$P(\sup_{m \le k \le T} \left| \frac{1}{\sqrt{k}} \sum\nolimits_{t=1}^{k} x_t \right| > M) \le \frac{\sigma^2}{M^2}(1 + \sum\nolimits_{k=m+1}^{T} \frac{1}{k}),$$

hence when $m = 1$, $\sup_{1 \le k \le T} \left| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} x_t \right| = O_p(\sqrt{\log T})$ as $T \to \infty$ since $\sum_{k=1}^{T} \frac{1}{k} - \log T$ converges to the Euler constant and when $m = [T\tau]$ for $\tau \in (0, 1)$, $\sup_{m \le k \le T} \left| \frac{1}{\sqrt{k}} \sum_{t=1}^{k} x_t \right| = O_p(1)$ as $T \to \infty$ since $\sum_{k=m+1}^{T} \frac{1}{k} = \sum_{k=1}^{T} \frac{1}{k} - \sum_{k=1}^{T\tau} \frac{1}{k} \to \log T - \log T\tau = \log \frac{1}{\tau}$. The last result also can be obtained from the functional central limit theorem.

Throughout the paper, $\|A\| = (trAA')^{\frac{1}{2}}$ denotes the Frobenius norm, $\xrightarrow{p}$ denotes convergence in probability, $\xrightarrow{d}$ denotes convergence in distribution, $vec(A)$ denotes the vectorization of matrix A, $r(A)$ denotes the rank of matrix A, $\delta_{NT} = \min\{\sqrt{N}, \sqrt{T}\}$, $(N,T) \to \infty$ denotes N and T going to infinity jointly.

3 EQUIVALENT REPRESENTATION AND ASSUMPTIONS

Since at least one of Λ_{01} and Λ_{02} is of full rank, we may assume Λ_{01} is of full rank in the following analysis without loss of generality. Due to symmetry, all results can be established similarly in case Λ_{02} is of full rank. When Λ_{01} is of full rank, the rank of the $N \times (r+q)$ matrix $\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{bmatrix}$ is between r and $r+q$. Suppose $\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{bmatrix}$ is of rank $r+q_1$, where $0 \le q_1 \le q$, then Λ_2 can be decomposed into $\Lambda_2 = \begin{bmatrix} \Lambda_{21} & \Lambda_{22} \end{bmatrix}$, where Λ_{21} is of dimension $N \times q_1$ and contains the columns in Λ_2 that are linearly independent of Λ_{01} and Λ_{22}. Λ_{22} is of dimension $N \times q_2$ and contains the columns in Λ_2 that are linear combinations of columns in $\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix}$ such that $\Lambda_{22} = \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix} Z$ for some $(r+q_1) \times q_2$ matrix Z. Therefore, $\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix}$ is of full rank $(r+q_1)$ and

$$\begin{aligned} \begin{bmatrix} \Lambda_0 & \Lambda_1 \end{bmatrix} &= \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix} A, \\ \begin{bmatrix} \Lambda_0 & \Lambda_2 \end{bmatrix} &= \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix} B, \end{aligned}$$

where $A = \begin{bmatrix} I_r \\ 0_{q_1 \times r} \end{bmatrix}$ and $B = \begin{bmatrix} I_{r-q} & 0_{(r-q)\times q_1} & \\ 0_{q\times(r-q)} & 0_{q\times q_1} & Z \\ 0_{q_1\times(r-q)} & I_{q_1} & \end{bmatrix}$. It follows that model (2) has the following equivalent representation with stable factor loadings:

$$\begin{aligned} X &= \begin{bmatrix} \begin{bmatrix} F_1^0 & F_1^1 \end{bmatrix} \begin{bmatrix} \Lambda_0 & \Lambda_1 \end{bmatrix}' \\ \begin{bmatrix} F_2^0 & F_2^1 \end{bmatrix} \begin{bmatrix} \Lambda_0 & \Lambda_2 \end{bmatrix}' \end{bmatrix} + E \\ &= \begin{bmatrix} \begin{bmatrix} F_1^0 & F_1^1 \end{bmatrix} (\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix} A)' \\ \begin{bmatrix} F_2^0 & F_2^1 \end{bmatrix} (\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix} B)' \end{bmatrix} + E \\ &= \begin{bmatrix} \begin{bmatrix} F_1^0 & F_1^1 \end{bmatrix} A' \\ \begin{bmatrix} F_2^0 & F_2^1 \end{bmatrix} B' \end{bmatrix} \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix}' + E. \end{aligned} \tag{4}$$

Next, define $G = (g_1, ..., g_T)' = \begin{bmatrix} \begin{bmatrix} F_1^0 & F_1^1 \end{bmatrix} A' \\ \begin{bmatrix} F_2^0 & F_2^1 \end{bmatrix} B' \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_{21} \end{bmatrix}$, then

$$X = G\Gamma' + E, \tag{5}$$

$$g_t = \begin{cases} Af_t, \ if \ 1 \le t \le [\tau_0 T] \\ Bf_t, \ if \ [\tau_0 T]+1 \le t \le T \end{cases}, \tag{6}$$

and we call $r+q_1$ the number of pseudo factors. Equivalent representation of model (2) was first formulated by Han and Inoue (2014). Here our representation is unified, generalizes and complements their result. Our representation is fairly general. The big break case discussed in Chen et al. (2014) corresponds to the case $q_1 = q$, while the type 1, type 2 and type 3 breaks discussed in Han and Inoue (2014) correspond to the cases $q_1 = q$, $q_1 = 0$ and $0 < q_1 < q$ respectively. The type 1 and type 2 changes discussed in Cheng et al. (2014) are also special cases of this representation. To ensure this equivalent representation is unique up to a rotation, it remains to show G is asymptotically full rank, i.e. $\frac{1}{T}\sum_{t=1}^{T} g_t g_t' \xrightarrow{p} \Sigma_G$ for some positive definite Σ_G. Define $\Sigma_F = E(f_t f_t')$, $\Sigma_{G,1} = E(g_t g_t')$ for $t \leq k_0$ and $\Sigma_{G,2} = E(g_t g_t')$ for $t > k_0$, then

$$\Sigma_{G,1} = A\Sigma_F A',\ \Sigma_{G,2} = B\Sigma_F B', \tag{7}$$

$$\Sigma_G = \tau_0 A\Sigma_F A' + (1-\tau_0)B\Sigma_F B'. \tag{8}$$

Proposition 1 *If $\tau_0 \in (0,1)$ and Σ_F is positive definite, Σ_G is positive definite.*

Proof. In Assumption 1 below, Σ_F is assumed to be positive definite, hence $A\Sigma_F A'$ and $B\Sigma_F B'$ are both positive semidefinite. For any $r+q_1$ dimensional vector v, if $v'\Sigma_G v = \tau_0 v' A\Sigma_F A' v + (1-\tau_0)v' B\Sigma_F B' v = 0$, it follows that $v'A\Sigma_F A'v = 0$ and $v'B\Sigma_F B'v = 0$. Again because Σ_F is positive definite, this implies $A'v = 0$ and $B'v = 0$. Plug in A, it follows that the first r elements of v are zero. Plug in B, it follows that the last q_1 elements of v are zero. These together imply that $v = 0$ and consequently Σ_G is positive definite. ■

For the case where Λ_{02} is of full rank, Λ_1 can be decomposed as $\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \end{bmatrix}$, where $\begin{bmatrix} \Lambda_0 & \Lambda_2 & \Lambda_{11} \end{bmatrix}$ is of full rank and $\Lambda_{12} = \begin{bmatrix} \Lambda_0 & \Lambda_2 & \Lambda_{11} \end{bmatrix} Z$ for some $(r+q_1)\times q_2$ matrix Z. Define $\Theta = \begin{bmatrix} \Lambda_0 & \Lambda_2 & \Lambda_{11} \end{bmatrix}$.

Our assumptions are as follows:

Assumption 1 *(1) $E\|f_t\|^4 < M < \infty$, $E(f_t f_t') = \Sigma_F$, Σ_F is positive definite, $\frac{1}{k_0}\sum_{t=1}^{k_0} f_t f_t' \xrightarrow{p} \Sigma_F$, $\frac{1}{T-k_0}\sum_{t=k_0+1}^{T} f_t f_t' \xrightarrow{p} \Sigma_F$, (2) $A\Sigma_F A' \neq B\Sigma_F B'$.*

Assumption 2 *$\|\lambda_{l,i}\| \leq \bar{\lambda} < \infty$ for $l = 0,1,2$, $\left\|\frac{1}{N}\Gamma'\Gamma - \Sigma_\Gamma\right\| \to 0$ for some positive definite matrix Σ_Γ or $\left\|\frac{1}{N}\Theta'\Theta - \Sigma_\Theta\right\| \to 0$ for some positive definite matrix Σ_Θ.*

Assumption 3 *There exists a positive constant $M < \infty$ such that:*

1 $E(e_{it}) = 0$, $E|e_{it}|^8 \leq M$, for all $i = 1,...,N$, and $t = 1,...,T$, *2* $E(\frac{e_s'e_t}{N})=\gamma_N(s,t)$, $|\gamma_N(s,s)|\leq M$ *for all* $s=1,...,T$, *also* $\frac{1}{T}\sum_{s=1}^{T}\sum_{t=1}^{T}|\gamma_N(s,t)|\leq M$,

3 $E(e_{it}e_{jt})=\tau_{ij,t}$ *with* $|\tau_{ij,t}|\leq\tau_{ij}$ *for some* τ_{ij} *and for all* $t=1,...,T$, *also* $\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N}|\tau_{ij}|\leq M$,

4 $E(e_{it}e_{js})=\tau_{ij,ts}$ *for* $i,j=1,...,N$, *and* $t,s=1,...,T$, *also* $\frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}|\tau_{ij,ts}|\leq M$,

5 For every $(t,s=1,...,T)$, $E\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^4\leq M$.

Assumption 4 $E(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{\sqrt{k_0}}\sum_{t=1}^{k_0}f_te_{it}\right\|^2)\leq M$ *and* $E(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{\sqrt{T-k_0}}\sum_{t=k_0+1}^{T}f_te_{it}\right\|^2)\leq M$.

Assumption 5 *There exists an* $M<\infty$ *such that for all* N *and* T *and for every* $t\leq T$ *and* $i\leq N$,

1 $\sum_{s=1}^{T}|\gamma_N(s,t)|\leq M$,

2 $\sum_{j=1}^{N}|\tau_{ji}|\leq M$.

Assumption 6 *The largest eigenvalue of* $\frac{1}{NT}EE'$ *is* $O_p(\frac{1}{\delta_{NT}^2})$.

Assumption 7 *The eigenvalues of* $\Sigma_G\Sigma_\Gamma$ *or* $\Sigma_G\Sigma_\Theta$ *are distinct.*

Assumption 8 *Define* $\epsilon_t=vec(f_tf_t'-\Sigma_F)$.*The data generating process of factors is such that the Hajek-Renyi inequality applies to the process* $\{\epsilon_t,t=1,...,k_0\}$, $\{\epsilon_t,t=k_0,...,1\}$, $\{\epsilon_t,t=k_0+1,...,T\}$ *and* $\{\epsilon_t,t=T,...,k_0+1\}$.

Assumption 9 $\frac{\log T}{N}\to 0$.

Assumption 10 *There exists* $M<\infty$ *such that:*

1 For every $s=1,...,T$, $E(\sup\limits_{k<k_0}\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^2)\leq M$,

$E(\sup\limits_{k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^2)\leq M$,

$E(\sup\limits_{k>k_0}\frac{1}{k-k_0}\sum_{t=k_0+1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^2)\leq M$,

$E(\sup\limits_{k\geq k_0}\frac{1}{T-k}\sum_{t=k+1}^{T}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^2)\leq M$,

2 $E(\sup\limits_{k<k_0}\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_ie_{it}\right\|^2)\leq M$,

$E(\sup\limits_{k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_ie_{it}\right\|^2)\leq M$,

$E(\sup\limits_{k>k_0}\frac{1}{k-k_0}\sum_{t=k_0+1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_ie_{it}\right\|^2)\leq M$,

$E(\sup\limits_{k\geq k_0}\frac{1}{T-k}\sum_{t=k+1}^{T}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_ie_{it}\right\|^2)\leq M$.

Assumptions 1-7 are either from or slight modification of Assumptions A-G in Bai (2003). Assumption 1(1) corresponds to Assumption A in Bai (2003) and should be satisfied within each regime. f_t can be dynamic and contain their lags. Assumption 1(2) enables the identification of the change point of the second moment matrix of g_t and is general enough to cover all patterns of factor loading break likely in practice. If $r(\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{bmatrix}) > r(\begin{bmatrix} \Lambda_0 & \Lambda_1 \end{bmatrix})$, then $A\Sigma_F A' \neq B\Sigma_F B'$. If $r(\begin{bmatrix} \Lambda_0 & \Lambda_1 & \Lambda_2 \end{bmatrix}) = r(\begin{bmatrix} \Lambda_0 & \Lambda_1 \end{bmatrix})$, then $A\Sigma_F A' = \Sigma_F \neq B\Sigma_F B'$ except for some very unlikely case, for example, all post-break factor loadings are -1 times the pre-break factor loadings. Note that here to simplify analysis, the second moment matrix of the factors is assumed to be stationary over time, since in general how to disentangle structural change in Σ_F from structural change in factor loadings is still unclear. Assumption 2 corresponds to Assumption B in Bai (2003) and implies that $\left\| \frac{1}{N}\Lambda_{01}'\Lambda_{01} - \Sigma_{\Lambda_{01}} \right\| \to 0$ and $\left\| \frac{1}{N}\Lambda_{02}'\Lambda_{02} - \Sigma_{\Lambda_{02}} \right\| \to 0$. Note that one of Λ_{01} and Λ_{02} is allowed to be degenerate. This allows for cases with disappearing or emerging factors. In addition, Λ_0 could contain a small change. As discussed in Bates et al. (2013), if $\Delta\lambda_{0,i} = \frac{1}{\sqrt{NT}}\kappa_i$ and $\|\kappa_i\| \leq \bar{\kappa} < \infty$ for all i, consistency of the estimated number of factors and the factors themselves will not be affected. For simplicity, we assume that Λ_0 is stable. Assumptions 3 and 5 are Assumptions C and E in Bai (2003), which allow for the temporal and cross-sectional dependence as well as heteroscedasticity. Assumption 4 corresponds to Assumption D in Bai (2003) and should be satisfied within each regime. This is implied by Assumptions 1 and 3 if the factors and the errors are independent. Assumption 6 is the key condition for identifying the number of factors and is implicitly assumed in Bai and Ng (2002) and required in almost all existing methods of determining the number of factors or the number of dynamic factors. For example, Onatski (2010) and Ahn and Horenstein (2013) assume $E = A\varepsilon B$, where ε is an i.i.d. $T \times N$ matrix and A and B characterize the temporal and cross-sectional dependence and heteroscedasticity. This is a sufficient but not necessary condition for Assumption 6. In this paper, Assumption 6 can be relaxed to "The largest eigenvalue of $\frac{1}{NT}EE'$ is $o_p(1)$", yet still allows consistent estimation of the number of factors. Assumption 7 corresponds to Assumption G in Bai (2003).

Assumption 8 strengthens Assumption 1(1) and imposes further requirement on the factor process. Instead of assuming a specific data generating process, here we only require that the Hajek-Renyi inequality is applicable to the second moment process of the factors, which incorporates i.i.d., martingale difference, martingale, mixingale and so on as special cases and renders Assumption 8 in its most general form. Assumption 10 imposes further constraints on the idiosyncratic error. Assumption 3(5) and Assumption F3 in Bai (2003) imply that the summands in Assumption 10 are uniformly $O_p(1)$. Assumption 10 strengthens this condition such that the supremum of the average process of these summands is $O_p(1)$. Nevertheless, Assumption 10 can be easily satisfied. If the functional CLT or the Hajek-Renyi inequality is applicable, Assumption 10 is directly implied by Assumption 3(5) and Assumption F3. Also note that stationarity is not assumed in Assumption 10. In rare cases, Assumption 10 is not satisfied, but we can still proceed with Assumption 9. Compared to $\frac{\sqrt{T}}{N} \to 0$, which is assumed in Chen et al. (2014), Han and Inoue (2014), Assumption 9 is significantly weaker and much easier to be satisfied since even when T is much larger than N, $\frac{\log T}{N}$ could still be very close to zero.

4 ESTIMATING THE CHANGE POINT

In this section, we discuss estimation of the change point with an unknown number of factors and where the factors are unobservable. First, we estimate the number of factors ignoring structural change using methods developed for the stable model, e.g., Bai and Ng (2002), Onatski (2010) and Ahn and Horenstein (2013). Define $\tilde{r}$ as the estimated number of factors, we will have $\lim_{(N,T)\to\infty} P(\tilde{r} = r + q_1) = 1$, since model (2) is observationally equivalent to model (4). Note that q_1 could be zero, since structural change does not necessarily lead to overestimating the number of factors. Using $\tilde{r}$, we then estimate the factors using the principal component method. This identifies the factors g_t. As noted in (7), the second moment matrix of g_t has a break at the point k_0. Hence, estimating change point of factor loadings can be converted to estimating change point of the second moment matrix of g_t. Although g_t is not directly observable, the principal component estimator $\tilde{g}_t$ is asymptotically close to $J'g_t$, where J is a rotation matrix and $J \xrightarrow{p} J_0$ as $(N,T) \to \infty$. Hence change point estimation using $\tilde{g}_t$ will be asymptotically equivalent to using $J_0 g_t$. It is easy to see that the second moment matrix of $J_0 g_t$ shares the same change point as that of g_t. Therefore, we proceed to estimate the pre-break and post-break second moment matrix of g_t using the estimated factors $\tilde{g}_t$. More specifically, for any $k > 0$, we split the sample into two subsamples and estimate the pre-break and post-break second moment matrix of g_t as

$$\begin{aligned} \tilde{\Sigma}_1 &= \frac{1}{k} \sum\nolimits_{t=1}^{k} \tilde{g}_t \tilde{g}_t', \\ \tilde{\Sigma}_2 &= \frac{1}{T-k} \sum\nolimits_{t=k+1}^{T} \tilde{g}_t \tilde{g}_t'. \end{aligned} \tag{9}$$

We then define the sum of squared residuals as

$$\tilde{S}(k) = \sum\nolimits_{t=1}^{k} [vec(\tilde{g}_t\tilde{g}_t' - \tilde{\Sigma}_1)]'[vec(\tilde{g}_t\tilde{g}_t' - \tilde{\Sigma}_1)] + \sum\nolimits_{t=k+1}^{T} [vec(\tilde{g}_t\tilde{g}_t' - \tilde{\Sigma}_2)]'[vec(\tilde{g}_t\tilde{g}_t' - \tilde{\Sigma}_2)], \tag{10}$$

and the least squares estimator of the change point as

$$\tilde{k} = \arg\min \tilde{S}(k). \tag{11}$$

Here we use $\tilde{S}(k)$ rather than $S(k)$ to emphasize that the sum of squared residuals is based on the estimated factors.

Remark 1 *The change point estimator also can be based on $\hat{g}_t$ instead of $\tilde{g}_t$, where $(\hat{g}_1, ..., \hat{g}_T)' = \hat{G} = \tilde{G}V_{NT} = (\tilde{g}_1, ..., \tilde{g}_T)'V_{NT}$ and V_{NT} is diagonal and contains the first $r + q_1$ largest eigenvalues of $\frac{1}{NT}XX'$ in decreasing order.*

In what follows, we establish the rate of convergence and the limiting distribution of the proposed estimator. The rate of convergence allows us to identify the number of pre-break and post-break factors as well as the factor space. Since $\lim_{(N,T)\to\infty} P(\tilde{r} = r + q_1) = 1$, estimation of the change point based on $\tilde{r}$ and the true number of pseudo factors $r + q_1$ is asymptotically equivalent, the strict proof is similar to footnote 5 in Bai (2003). Therefore, we can treat the number of pseudo factors $r + q_1$ as known in studying the asymptotic properties of our change point estimator.

Define $\tilde{\tau} = \tilde{k}/T$ as the estimated change fraction, we first show that $\tilde{\tau}$ is consistent.

Proposition 2 *Under Assumptions 1-8 and 9 or 10, $\tilde{\tau} - \tau_0 = o_p(1)$.*

This proposition is important for theoretical purposes. In fact, in this paper it serves as a first step in proving Theorem 1. Here we sketch the main steps of the proof. To show $\tilde{\tau} - \tau_0 = o_p(1)$, we need to show that for any $\epsilon > 0$ and any $\eta > 0$, $P(|\tilde{\tau} - \tau_0| > \eta) < \epsilon$ holds for sufficiently large N and T . For a given η, define $D = \{k : (\tau_0 - \eta)T \le k \le (\tau_0 + \eta)T\}$, then we need to show that $P(\tilde{k} \in D^c) < \epsilon$ holds for sufficiently large N and T. By the definition of $\tilde{k}$, we know that $\tilde{S}(\tilde{k}) - \tilde{S}(k_0) \le 0$. If $\tilde{k} \in D^c$, then $\min_{k\in D^c} \tilde{S}(k) - \tilde{S}(k_0) \le 0$. It follows that $P(\tilde{k} \in D^c) \le P(\min_{k\in D^c} \tilde{S}(k) - \tilde{S}(k_0) \le 0)$. Next, as proved in the Appendix, $P(\min_{k\in D^c} \tilde{S}(k) - \tilde{S}(k_0) \le 0) = P(\min_{k\in D^c} \frac{\tilde{S}(k)-\tilde{S}(k_0)}{|k-k_0|} \le 0)$. Hence it suffices to show that $P(\min_{k\in D^c} \frac{\tilde{S}(k)-\tilde{S}(k_0)}{|k-k_0|} \le 0) < \epsilon$ for sufficiently large N and T. We decompose $\frac{\tilde{S}(k)-\tilde{S}(k_0)}{|k-k_0|}$ into two parts. One is nonrandom and positive. The other one is random and uniformly $o_p(1)$ for $k \in D^c$, and hence dominated by the nonrandom part. This finishes the proof.

Using the same strategy, we can also prove that for any $\epsilon > 0$ and $\eta > 0$, there exist an $M > 0$ such that $P(\min_{k\in D_M} \frac{\tilde{S}(k)-\tilde{S}(k_0)}{|k-k_0|} \le 0) < \epsilon$ for sufficiently large N and T, where $D_M = \{k : k \in D, |k - k_0| > M\}$. This together with Proposition 2 implies:

Theorem 1 *Under Assumptions 1-8 and 9 or 10,* $\tilde{k} - k_0 = O_p(1)$.

This theorem implies that the difference between the estimated change point and the true change point is stochastically bounded. This is quite strong since the possible change point is narrowed to a bounded set no matter how large T is. Although $\tilde{k}$ is still inconsistent, $\tilde{k} - k_0 = O_p(1)$ is good enough for many purposes, especially, for consistent estimation of the number of pre-break and post-break factors and consistent estimation of the pre-break and post-break factor space, which will be discussed in the next two sections.

Although in the current setup, N goes to infinity jointly with T, our result is different from that of Bai (2010), in which he shows that $\lim_{(N,T)\to\infty} P(\tilde{k} = k_0) = 1$. Our result is similar to the univariate case, e.g., Bai (1994, 1997), Bai and Perron (1998). This is because $\tilde{k}$ is based on the sum of squared residuals of $g_t g_t'$, which is a fixed dimension multivariate process, not on the sum of squared residuals of x_t itself. The latter is infeasible because the factors are unobservable.

Theorem 1 also implies a nondegenerate limiting distribution of $\tilde{k}$. Define

$$\begin{aligned} y_t &= vec(J_0' g_t g_t' J_0 - \Sigma_1) \text{ for } t \leq k_0, \\ y_t &= vec(J_0' g_t g_t' J_0 - \Sigma_2) \text{ for } t > k_0, \end{aligned} \tag{12}$$

where $J_0 = \Sigma_\Gamma^{\frac{1}{2}} \Phi V^{-\frac{1}{2}}$, V is the diagonal matrix of eigenvalues of $\Sigma_\Gamma^{\frac{1}{2}} \Sigma_G \Sigma_\Gamma^{\frac{1}{2}}$, Φ is the corresponding eigenvector matrix. $\Sigma_1 = J_0' \Sigma_{G,1} J_0$ and $\Sigma_2 = J_0' \Sigma_{G,2} J_0$ are the pre-break and post-break means of $J_0' g_t g_t' J_0$. The limiting distribution of $\tilde{k}$ is as follows:

Theorem 2 *Under Assumptions 1-8 and 9 or 10,* $\tilde{k} - k_0 \xrightarrow{d} \arg\min W(l)$, *where*

$$\begin{aligned} W(l) &= -l \left\| \Sigma_2 - \Sigma_1 \right\|^2 - 2 \sum\nolimits_{t=k_0+l}^{k_0-1} [vec(\Sigma_2 - \Sigma_1)]' y_t \textit{ for } l = -1, -2, ..., \\ W(l) &= 0 \textit{ for } l = 0, \\ W(l) &= l \left\| \Sigma_2 - \Sigma_1 \right\|^2 - 2 \sum\nolimits_{t=k_0+1}^{k_0+l} [vec(\Sigma_2 - \Sigma_1)]' y_t \textit{ for } l = 1, 2, ... \end{aligned} \tag{13}$$

This theorem shows that if y_t is independent over t, then $W(l)$ is a two-sided random walk. Note that y_t is not assumed to be stationary. By definition, if f_t is stationary, then g_t and hence y_t is stationary within each regime. In this case $\sum_{t=k_0+l}^{k_0-1}$ and $\sum_{t=k_0+1}^{k_0+l}$ can be replaced by $\sum_{t=l}^{-1}$ and $\sum_{t=1}^{l}$. In previous change point estimation studies, there are two frameworks for deriving the limiting distribution. The first one assumes the magnitude of the break v_T is fixed, and the second one assumes $v_T \to 0$ but $T v_T^2 \to \infty$. In the second framework, $\tilde{k} - k_0 = O_p(\frac{1}{v_T^2})$ and

$\frac{1}{A}v_T^2(\tilde{k}-k_0) \xrightarrow{d} \arg\min[-|l|+2B(l)]$, where $B(l)$ is a two sided Brownian motion on the real line and A is a constant. The shrinking break assumption is indeed imposed for mathematical convenience to make the limiting distribution independent of the underlying DGP. However, in the current setup, the break magnitude $\|\Sigma_2 - \Sigma_1\|$ is fixed and it is unreasonable to assume $\|\Sigma_2 - \Sigma_1\| \to 0$ as $T \to \infty$. Bai (2010) also considers a fixed magnitude for the break. His Theorem 4.2 shows $\tilde{k} - k_0 \xrightarrow{d} \arg\min[|l|\,\lambda + 2\sqrt{\phi}W(l)]$, where $W(l)$ is a two sided standard Gaussian random walk, with l and ϕ denoting two parameters. The difference between our result and Bai (2010) is that our random walk is not necessarily Gaussian. This is because the dimension of y_t, $(r+q_1)^2$, is fixed and y_{jt} and y_{kt} are not independent for $j \neq k$. In contrast, in Bai (2010), the dimension of e_t, N, goes to infinity and e_{jt} and e_{kt} are independent for $j \neq k$ so that the CLT applies to the weighted sum of e_{it}. Although the limiting distribution in Theorem 2 is not free of the true DGP, we can still construct a feasible confidence interval if we alternatively follow the smoothed least squares method in Seo and Linton (2007). Other approaches, e.g., Elliott and Muller (2007), are also promising. We leave the detailed study for future research.

Remark 2 *In some special cases, the limiting distribution of $\tilde{k} - k_0$ is one-sided, concentrating on $l \geq 0$. For example, if Λ_0, Λ_1 and $\Lambda_2 - \Lambda_1$ are orthogonal to each other and the factors are also orthogonal with each other, then $[vec(\Sigma_2 - \Sigma_1)]'y_t = 0$ for all $t < k_0$. It follows that $W(l) > W(0)$ for all $l < 0$, hence $\arg\min W(l) \geq 0$.*

Remark 3 *Proposition 2, Theorem 1 and Theorem 2 hold with either Assumption 9 or 10, but we do not need both. Usually Assumption 10 is satisfied. In this case, there is no restriction on the relative speed of N and T going to infinity. Even when Assumption 10 is violated, our results only require $\frac{\log T}{N} \to 0$, which can be easily satisfied.*

A significant advantage of our change point estimator is that even when the estimated number of pseudo factors $\tilde{r}$ is different from $r+q_1$, $\tilde{k}$ still performs well. This advantage improves the finite sample performance since in practice $\tilde{r}$ is likely to be different from $r+q_1$.

In case $\tilde{r}$ is fixed at some positive integer $m < r+q_1$, we have the following result:

Corollary 1 *For any positive integer $m < r+q_1$ and change point estimation based on $\tilde{r} = m$, with J_0 replaced by J_0^m which is of dimension $(r+q_1)\times m$ and contains the first m columns of J_0, and $J_0^{m\prime}\Sigma_{G,1}J_0^m \neq J_0^{m\prime}\Sigma_{G,2}J_0^m$, Proposition 2, Theorem 1 and Theorem 2 still hold.*

In case $\tilde{r}$ is fixed at some positive integer $m > r+q_1$, although we can not prove the corresponding result, our simulation results show that as long as m is not far away from $r+q_1$, $\tilde{k}$ performs quite well.

5 DETERMINING THE NUMBER OF FACTORS

In this section, we study how to consistently estimate the number of factors in the presence of structural instability in the factor loadings or the number of factors themselves. We first relax the sufficient condition proposed by Bates et al. (2013) for the consistent estimation of the number of factors in the presence of structural change using the Bai and Ng (2002) information criteria. The condition they propose is $\frac{1}{N}\|\Delta\|^2 = O(\frac{1}{\delta_{NT}^2})$, where Δ is the matrix of factor loading breaks. In the current setup, $\Delta = \Lambda_2 - \Lambda_1$. We show, in the following proposition, that their condition can be relaxed to $\frac{1}{N}\|\Delta\|^2 = O(\frac{1}{\delta_{NT}^c})$ for some $c > 0$.

Proposition 3 *In the presence of a single common break in factor loadings, the estimator of the number of factors using the Bai and Ng (2002) information criteria is still consistent if* $\frac{1}{N}\|\Delta\|^2 = O(\frac{1}{\delta_{NT}^c})$ *for some* $c > 0$*,* $g(N,T) \to 0$ *and* $\delta_{NT}^c g(N,T) \to \infty$*, where* $g(N,T)$ *is the penalty function.*

The formal proof is in the Appendix. This proposition complements Theorem 3 below. Note that c can be arbitrarily close to zero, hence our condition is much weaker than that of Bates et al. (2013). The intuition behind our result is that the change in factor loadings can be treated as an extra error term, and as long as $c > 0$, the first r largest eigenvalues of XX' are still separated from the rest. By adjusting the speed at which the penalty function goes to infinity accordingly, the number of factors can still be consistently determined. This idea is also used in Amengual and Watson (2007) to determine the number of dynamic factors.

Some caveats are the following: When c is less than two, the magnitude of this extra error term becomes large. To outweigh the error term, the speed at which the penalty function $g(N,T)$ goes to zero has to be slower than the speed at which $\frac{1}{N}\|\Delta\|^2$ goes to zero, so that $\frac{g(N,T)}{\frac{1}{N}\|\Delta\|^2} \to \infty$. This may be problematic in real applications, since when c is close to zero, not all factors are necessarily strong enough to outweigh the extra noise brought by the factor loadings breaks. In addition, the above result is not applicable for the case where $\frac{1}{N}\|\Delta\|^2 = O(1)$, nor the case where the number of factors also change. While $\frac{1}{N}\|\Delta\|^2 = O(\frac{1}{\delta_{NT}^c})$ approximates a small structural change, $\frac{1}{N}\|\Delta\|^2 = O(1)$ approximates a large structural change.

When Proposition 3 is not applicable, we are still able to identify the number of factors. Based on the estimated change point $\tilde{k}$, we split the sample into two subsamples, and then use each subsample to estimate the number of pre-break and post-break factors. Let $\tilde{r}_1$ and $\tilde{r}_2$ be the estimated number of pre-break and post-break factors using the method in Bai and Ng (2002). We have the following result:

Theorem 3 *Under Assumptions 1-8 and 9 or 10,* $\lim_{(N,T)\to\infty} P(\tilde{r}_1 = r_1) = 1$ *and* $\lim_{(N,T)\to\infty} P(\tilde{r}_2 = r_2) = 1$*, where* r_1 *and* r_2 *are numbers of pre-break and post-break factors, respectively.*

Theorem 3 implies that slight misspecification of the change point does not affect the consistency of the estimated number of pre-break and post-break factors. Meanwhile, Theorem 1 guarantees that the change point can only be slightly misspecified. In fact, our simulations show that even for finite sample performance of the estimated number of pre-break and post-break factors, estimators based on $\tilde{k}$ and k_0 have almost the same performance. Therefore, Theorem 3 together with Theorem 1 identify model (2) and provide the basis for subsequent estimation of model (2) and inference based on model (2).

As in Proposition 3, the change in factor loadings can be treated as an extra error term and as long as the magnitude of this extra error is small, the first r largest eigenvalues of XX' are still separated from the rest. Unlike Proposition 3 and Bates et al. (2013), in which ad hoc conditions of small break magnitude are imposed, here we allow the magnitude of structural change in each observation to be large. The key observation behind this theorem is that $\tilde{k} - k_0 = O_p(1)$ implies in each subsample the number of observations with factor loading break is bounded in probability no matter how large T is, hence for large T the average magnitude of structural change in each subsample is still small even though the magnitude of structural change in each observation is large.

Consider the consistency of $\tilde{r}_1$. Due to symmetry, the consistency of $\tilde{r}_2$ can be established similarly. The estimator of the number of pre-break factors $\tilde{r}_1$ is based on the pre-break subsample $t = 1, ..., \tilde{k}$. What we need to show is: for any $\epsilon > 0$, $P(\tilde{r}_1 \neq r_1) < \epsilon$ for large (N,T). Based on $\left|\tilde{k} - k_0\right| = O_p(1)$, we have for any $\epsilon > 0$, there exists $M > 0$ such that $P(\left|\tilde{k} - k_0\right| > M) < \epsilon$ for all (N,T). Based on this M, $P(\tilde{r}_1 \neq r_1)$ can be decomposed as

$$P(\tilde{r}_1 \neq r_1) = P(\tilde{r}_1 \neq r_1, \left|\tilde{k} - k_0\right| > M) + P(\tilde{r}_1 \neq r_1, k_0 - M \leq \tilde{k} \leq k_0) + P(\tilde{r}_1 \neq r_1, k_0 + 1 \leq \tilde{k} \leq k_0 + M).$$

The first term is less than $P(\left|\tilde{k}-k_0\right|>M)$, hence less than ϵ for all (N,T). The second term can be further decomposed as

$$P(\tilde{r}_1 \neq r_1, k_0 - M \leq \tilde{k} \leq k_0) = \sum\nolimits_{k=k_0-M}^{k_0} P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k),$$

where $P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k)$ denotes the joint probability of $\tilde{k} = k$ and $\tilde{r}_1(k) \neq r_1$ and $\tilde{r}_1(k)$ denotes the estimated number of pre-break factors using subsample $t = 1, ..., k$. Obviously, $P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k) \leq P(\tilde{r}_1(k) \neq r_1)$, hence the second term is less than $\sum_{k=k_0-M}^{k_0} P(\tilde{r}_1(k) \neq r_1)$. Furthermore, the factor loadings in the pre-break subsample are stable when $k < k_0$ and for $k \in [k_0 - M, k_0]$, $k \to \infty$ at the same speed as k_0, hence we have for each $k \in [k_0 - M, k_0]$, $P(\tilde{r}_1(k) \neq r_1) \leq \frac{\epsilon}{M+1}$ for large (N,T). The second term is therefore less than $\sum_{k=k_0-M}^{k_0} \frac{\epsilon}{M} = \epsilon$ for large (N,T).

The argument for the second term also applies to the third term, except for some modifications. First, the third term can be decomposed similarly as

$$P(\tilde{r}_1 \neq r_1, k_0 + 1 \leq \tilde{k} \leq k_0 + M) = \sum\nolimits_{k=k_0+1}^{k_0+M} P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k) \leq \sum\nolimits_{k=k_0+1}^{k_0+M} P(\tilde{r}_1(k) \neq r_1),$$

hence it remains to show for each $k \in [k_0 + 1, k_0 + M]$, $P(\tilde{r}_1(k) \neq r_1) \leq \frac{\epsilon}{M}$ for large (N,T). Unlike the second term, when $k \in [k_0 + 1, k_0 + M]$ the factor loadings of the pre-break subsample $t = 1, ..., k$ has a break at $t = k_0$, hence results already established for the stable model are not directly applicable. Nevertheless, the number of observations with factor loading break, $k - k_0$, is bounded by M. Hence in estimating the number of factors, these observations will be dominated by the observations $t = 1, ...k_0$, as $k_0 = [\tau_0 T] \to \infty$. The formal proof is in the appendix.

Remark 4 *Theorem 3 together with Theorem 1 also shed some light on the identification of dynamic factor models with structural instability. Amengual and Watson (2007), Bai and Ng (2007) and Hallin and Liska (2007) study the question of determining the number of dynamic factors in a large factor model. When the model has structural instability, these methods, however, are no longer applicable. Our estimation procedure provides a promising solution.*

Remark 5 $\tilde{r}_1$ *and* $\tilde{r}_2$ *could be based on the method proposed by Ahn and Horenstein (2013). Their method is also based on eigenvalue separation and in the proof of Theorem 3 we show that the extra error term brought by a change in factor loadings does not affect the eigenvalue separation, nor the speed of this eigenvalue separation. Therefore,* $\tilde{r}_1$ *and* $\tilde{r}_2$ *based on Ahn and Horenstein (2013) should be consistent.*

Remark 6 *In principle, $\tilde{r}_1$ and $\tilde{r}_2$ could also be based on the method proposed by Onatski (2010). However, Onatski (2010) requires calibration of the threshold, δ, which becomes more complicated with the extra error term introduced by the change in factor loadings.*

6 ESTIMATING THE FACTOR SPACE

In this section, we discuss how to estimate the pre-break and post-break factor space. As in last section, we split the sample into two subsamples based on the change point estimator $\tilde{k}$, and then use each subsample to estimate the pre-break and post-break factor space. For each possible sample split k, define $X(k) = (x_1, ..., x_k)'$, $F_1(k) = (f_1, ..., f_k)'$ and $F_2(k) = (f_{k+1}, ..., f_T)'$. Let u be any prespecified number of pre-break factors, which does not necessarily equal r_1. The principal component estimator of the pre-break factors and factor loadings are obtained by solving $V(u) = \min \frac{1}{Nk} \sum_{t=1}^{k} \sum_{i=1}^{N} (x_{it} - f_t' \lambda_i)^2$. Since the true factors can be identified only up to a rotation, the normalization condition has to be imposed to uniquely determine the solution, and based on different normalization conditions there are two solutions. For the first one, the estimated factors, $\tilde{F}_1^u(k)$, equal $\sqrt{T}$ times the eigenvectors corresponding to the first u largest eigenvalues of $\frac{1}{Nk} X(k) X'(k)$ and $\tilde{\Lambda}_1^u(k) = \frac{1}{k} X'(k) \tilde{F}_1^u(k)$ are the corresponding estimated factor loadings. For the second one, the estimated factor loadings, $\bar{\Lambda}_1^u(k)$, equal $\sqrt{N}$ times the eigenvectors corresponding to the first u largest eigenvalues of $\frac{1}{Nk} X'(k) X(k)$ and $\bar{F}_1^u(k) = \frac{1}{N} X(k) \bar{\Lambda}_1^u(k)$ are the corresponding estimated factors. Following Bai and Ng (2002), we define the rescaled estimator $\hat{F}_1^u(k) = \bar{F}_1^u(k) [\frac{1}{k} \bar{F}_1^{u\prime}(k) \bar{F}_1^u(k)]^{\frac{1}{2}}$. The estimator of the post-break factors $\hat{F}_2^v(k)$ can be obtained similarly based on the post-break subsample, where v is the prespecified number of post-break factors. Next, define $H_1^u(k) = \frac{\Lambda_{01}' \Lambda_{01}}{N} \frac{F_1'(k) \tilde{F}_1^u(k)}{k}$ and $H_2^v(k) = \frac{\Lambda_{02}' \Lambda_{02}}{N} \frac{F_2'(k) \tilde{F}_2^v(k)}{T-k}$. Let $\hat{f}_t^u(\tilde{k})$ and $\hat{f}_t^v(\tilde{k})$ be the estimated factors based on change point estimator $\tilde{k}$ for $t \leq \tilde{k}$ and $t > \tilde{k}$ respectively, we have the following theorem:

Theorem 4 *Under Assumptions 1-8 and 9 or 10,* $\frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ *and* $\frac{1}{T-\tilde{k}} \sum_{t=\tilde{k}+1}^{T} \left\| \hat{f}_t^v(\tilde{k}) - H_2^{v\prime}(\tilde{k}) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$.

The intuition of this theorem is similar to that of Theorem 3. Just consider estimation of the pre-break factor space. What we need to show is: for any $\epsilon > 0$, there exist $C > 0$ such that $P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C) < \epsilon$ for all (N, T). As in Theorem 3, we can choose $M > 0$ such that $P(\left| \tilde{k} - k_0 \right| > M) < \frac{\epsilon}{2}$ for the given ϵ. Based on this M,

$P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C)$ can be decomposed as

$$\begin{aligned} P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C) &= P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C, \left| \tilde{k} - k_0 \right| > M) \\ &+ \sum_{k=k_0-M}^{k_0} P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C, \tilde{k} = k) \\ &+ \sum_{k=k_0+1}^{k_0+M} P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C, \tilde{k} = k). \end{aligned}$$

The first term is less than $P(\left| \tilde{k} - k_0 \right| > M)$, hence less than ϵ for all (N, T). The second term is less than $\sum_{k=k_0-M}^{k_0} P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C)$. Furthermore, Theorem 1 of Bai and Ng (2002) is applicable for each $k \in [k_0 - M, k_0]$ for reasons similar to the second term of Theorem 3, i.e. the factor loadings in the pre-break subsample are stable when $k < k_0$ and for $k \in [k_0 - M, k_0]$, $k \to \infty$ at the same speed as k_0. The second term is therefore less than $\sum_{k=k_0-M}^{k_0} \frac{\epsilon}{M+1} = \epsilon$ for large (N, T). The third term is less than $\sum_{k=k_0+1}^{k_0+M} P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C)$. Similar to the third term of Theorem 3, although the factor loadings of the pre-break subsample has a break at $t = k_0$ when $k \in [k_0 + 1, k_0 + M]$, the number of observations with factor loading break is bounded by M. In estimating the factor space, these observations will also be dominated by the observations $t = 1, ...k_0$, as $k_0 = [\tau_0 T] \to \infty$. The formal proof is in the appendix.

Remark 7 *Theorem 4 is based on arbitrarily u and v rather than $\tilde{r}_1$ and $\tilde{r}_2$, the estimated number of pre-break and post-break factors. On the other hand, $\tilde{r}_1$ and $\tilde{r}_2$ are based directly on eigenvalue separation, without using consistency of the estimated pre-break and post-break factor space. Hence, Theorem 4 and Theorem 3 are independent with each other. Alternatively, we can choose $u = \tilde{r}_1$ and $v = \tilde{r}_2$. Since $\tilde{r}_1$ and $\tilde{r}_2$ are consistent, this is asymptotically equivalent to the case in which r_1 and r_2 are known. The same argument was used by Bai (2003) for deriving the limiting distribution of the estimated factors. When r_1 and r_2 are known and under Assumptions 1-8 and 9 or 10, we have $\frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t(\tilde{k}) - H_1'(\tilde{k}) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ and $\frac{1}{T-\tilde{k}} \sum_{t=\tilde{k}+1}^{T} \left\| \hat{f}_t(\tilde{k}) - H_2'(\tilde{k}) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$.*

Theorem 4 implies that our estimator of the factor space is mean square consistent within each regime and the convergence rate is the same as that obtained by Bai and Ng (2002) for the stable model. Consistent estimation of the factor space has proved to be crucial in many cases, including forecasting and factor augmented regressions. Note that the convergence rate $O_p(\frac{1}{\delta_{NT}^2})$ plays a crucial role in eliminating the effect of using estimated factors, for which merely consistency is not enough. Bates et al. (2013) shows that if we ignore the structural change, consistency of the estimated factor space requires $\frac{1}{N}\|\Delta\|^2 = o(1)$ while to guarantee the convergence rate $O_p(\frac{1}{\delta_{NT}^2})$ of the estimated factor space, it requires $\frac{1}{N}\|\Delta\|^2 = O(\frac{1}{\delta_{NT}})$. While reasonable for small break, these two conditions especially the latter are not suitable for large break, which as discussed in Banerjee, Marcellino and Masten (2008), is the most likely reason behind the worsening factor-based forecasts. In contrast, our result allows for large break, hence improves and complements Bates et al. (2013).

More specifically, consider the following diffusion index forecasting model

$$y_{t+h} = \alpha' f_t + \beta' W_t + \epsilon_{t+h}, \tag{14}$$

where the factors are unobservable and estimated from model (1). Here for illustration we have assumed that the forecasting equation (14) itself is stable. If the change point k_0 were known, we could split the sample and estimate the pre-break and post-break factors separately. Accordingly, y_{t+h} has the following transformation:

$$y_{t+h} = \begin{cases} \alpha'[H_1'(k_0)]^{-1}H_1'(k_0)f_t + \beta' W_t + \epsilon_{t+h}, \ if \ 1 \leq t \leq k_0 \\ \alpha'[H_2'(k_0)]^{-1}H_2'(k_0)f_t + \beta' W_t + \epsilon_{t+h}, \ if \ k_0+1 \leq t \leq T \end{cases}.$$

Define $\alpha_1' = \alpha'[H_1'(k_0)]^{-1}$, $\alpha_2' = \alpha'[H_2'(k_0)]^{-1}$and $\delta = (\alpha_1', \alpha_2', \beta')'$, it follows that

$$y_{t+h} = \begin{cases} \alpha_1'\hat{f}_t(k_0) + \beta' W_t + \epsilon_{t+h} + \xi_t, \ if \ 1 \leq t \leq k_0 \\ \alpha_2'\hat{f}_t(k_0) + \beta' W_t + \epsilon_{t+h} + \xi_t, \ if \ k_0+1 \leq t \leq T \end{cases},$$

where $\xi_t = \begin{cases} -\alpha_1'[\hat{f}_t(k_0) - H_1'(k_0)f_t], \ if \ 1 \leq t \leq k_0 \\ -\alpha_2'[\hat{f}_t(k_0) - H_2'(k_0)f_t], \ if \ k_0+1 \leq t \leq T \end{cases}$. The feasible prediction is $\hat{y}_{T+h|T} = \hat{\alpha}_2'\hat{f}_T(k_0) + \hat{\beta}' W_T$, where $\hat{\alpha}_2$ and $\hat{\beta}$ are the least squares estimators obtained from regressing y_{t+h} on $\hat{f}_t(k_0)$ and W_t, $t = 1, ..., T-h$. Since factor loadings are stable within each regime, Theorem 1 in Bai and Ng (2002) is applicable, hence we have $\frac{1}{k_0}\sum_{t=1}^{k_0}\left\|\hat{f}_t(k_0) - H_1'(k_0)f_t\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ and $\frac{1}{T-k_0}\sum_{t=k_0+1}^{T}\left\|\hat{f}_t(k_0) - H_2'(k_0)f_t\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$. Based on this and following Bai and Ng (2006), it is not difficult to show that ξ_t can be ignored asymptotically given$\frac{\sqrt{T}}{N} \to 0$ and $\hat{\delta}$ and $\hat{y}_{T+h|T}$ are consistent and asymptotically normal.

If the change point k_0 is unknown and estimated as in Section 4, $\tilde{k} - k_0 = O_p(1)$ guarantees Theorem 4 are as good as $\frac{1}{k_0}\sum_{t=1}^{k_0}\left\|\hat{f}_t(k_0) - H_1'(k_0)f_t\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ and $\frac{1}{T-k_0}\sum_{t=k_0+1}^{T}\left\|\hat{f}_t(k_0) - H_2'(k_0)f_t\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$, hence consistency and asymptotic normality of $\hat{\delta}$ and $\hat{y}_{T+h|T}$ still can be established given$\frac{\sqrt{T}}{N} \to 0$. The intuition is similar to that of Theorem 3 and Theorem 4. $\tilde{k} - k_0 = O_p(1)$ guarantees the number of miscategorized observations, $\tilde{k} - k_0$, is bounded in probability. $\hat{\delta}$ and $\hat{y}_{T+h|T}$ depend on the whole sample, hence these observations will be dominated by the correctly categorized observations which is of size $O(T)$. The detailed proof as well as other related issues will be studied in future research.

7 SIMULATIONS

7.1 DESIGN

Our design roughly follows that of Bates et al. (2013), with the focus switching from small change to large change and from forecasting to estimating the whole model, i.e. estimating the change point, the number of pre-break and post-break factors and the pre-break and post-break factor spaces.

The data is generated as follows:

$$x_{it} = \begin{cases} f_{0,t}'\lambda_{0,i} + f_{1,t}'\lambda_{1,i} + \sqrt{\theta_1}e_{i,t}, & if\ 1 \le t \le [\tau_0 T] \\ f_{0,t}'\lambda_{0,i} + f_{1,t}'\lambda_{2,i} + \sqrt{\theta_2}e_{i,t}, & if\ [\tau_0 T]+1 \le t \le T \end{cases} \text{ for } i = 1, ..., N \text{ and } t = 1, ..., T.$$

As discussed in Section 2, in case the number of pre-break and post-break factors is r_1 and r_2 respectively, with $r = \max\{r_1, r_2\}$, f_t and λ_i are always r dimensional vectors. If $r_1 < r_2$, the last $r_2 - r_1$ elements of $\lambda_{1,i}$ are zeros while if $r_1 > r_2$, the last $r_1 - r_2$ elements of $\lambda_{2,i}$ are zeros. θ_1 and θ_2 control the magnitude of noise and here we take $\theta_1 = r_1, \theta_2 = r_2$.

The factors are generated as follows:

$$f_{t,p} = \rho f_{t-1,p} + u_{t,p} \text{ for } t = 2, ..., T \text{ and } p = 1, ..., r,$$

where $u_{t,p}$ is i.i.d. $N(0,1)$ for $t = 2, ..., T$ and $p = 1, ..., r$. For $t = 1$, $f_{1,p}$ is i.i.d. $N(0, \frac{1}{1-\rho^2})$ for $p = 1, ..., r$ so that factors have stationary distributions. The scalar ρ captures serial correlation of factors.

The idiosyncratic errors are generated as follows:

$$e_{i,t} = \alpha e_{i,t-1} + v_{i,t} \text{ for } i = 1, ..., N \text{ and } t = 2, ..., T.$$

The processes $\{u_{t,p}\}$ and $\{v_{i,t}\}$ are mutually independent with $v_t = (v_{1,t}, ..., v_{N,t})'$ being i.i.d. $N(0, \Omega)$ for $t = 2, ..., T$. For $t = 1$, $e_{\cdot,1} = (e_{1,1}, ..., e_{N,1})'$ is $N(0, \frac{1}{1-\alpha^2}\Omega)$ so that the idiosyncratic errors have stationary distributions. The scalar α captures the serial correlation of the idiosyncratic errors. As in Bates et al. (2013), $\Omega_{ij} = \beta^{|i-j|}$ captures the cross-sectional dependence of the idiosyncratic errors.

We consider three different ways of generating factor loadings corresponding to three different representative setups. The first setup allows both change in the number of factors and partial change in the factor loadings, with $(r_1, r_2) = (3, 5)$ and one factor having stable loadings. In this case, $\lambda_{0,i}$ is independent $N(0, x_i(R_i^2))$ across i, both $\lambda_{1,i}$ and $\lambda_{2,i}$ are four dimensional vectors, the first two elements of $\lambda_{1,i}$ are independent $N(0, x_i(R_i^2)I_2)$ across i and the last two elements of $\lambda_{1,i}$ are zeros while $\lambda_{2,i}$ is independent $N(0, x_i(R_i^2)I_4)$ across i, hence the number of pseudo factors in the equivalent representation is $r_1 + r_2 - 1 = 7$. The scalar $x_i(R_i^2)$ is determined so that the regression R^2 of series i is equal to R_i^2[1]. The second setup allows only change in the number of factors, with $(r_1, r_2) = (3, 5)$ and three factors having stable loadings. In this case, $\lambda_{0,i}$ is independent $N(0, x_i(R_i^2)I_3)$ across i, both $\lambda_{1,i}$ and $\lambda_{2,i}$ are two dimensional vectors, $\lambda_{1,i}$ are zeros while $\lambda_{2,i}$ is independent $N(0, x_i(R_i^2)I_2)$ across i, hence the number of pseudo factors is 5. The third setup allows only partial change in the factor loadings, with $(r_1, r_2) = (3, 3)$ and one factor having stable loadings. In this case, $\lambda_{0,i}$ is independent $N(0, x_i(R_i^2))$ across i, both $\lambda_{1,i}$ and $\lambda_{2,i}$ are two dimensional vectors, $\lambda_{1,i}$ is independent $N(0, x_i(R_i^2)I_2)$ across i while $\lambda_{2,i} = (1-a)\lambda_{1,i} + \sqrt{2a - a^2}d_i$, where $a \in [0, 1]$ and d_i is independent $N(0, x_i(R_i^2)I_2)$ across i, hence the number of pseudo factors is 5 except for $a = 0$. The scalar a captures the magnitude of factor loading changes, with the the ratio of mean square changes in the factor loadings to the pre-break factor loadings being equal to $\frac{4a}{3}$. We consider $a = 0.2$, 0.6 and 1, which correspond to small, medium and large changes respectively. Finally, all factor loadings are independent of the factors and the idiosyncratic errors.

For each setup, we consider the benchmark DGP with $(\rho, \alpha, \beta) = (0, 0, 0)$ and homogeneous R^2 and the more empirically relevant DGP with $(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$ and heterogeneous R^2. For homogeneous R^2, $R_i^2 = 0.5$ for all i, which is also considered in Bai and Ng (2002), Ahn and Horenstein (2013) (to name a few) as benchmark case in evaluating estimators of number of factors. For heterogeneous R^2, R_i^2 is drawn from $U(0.2, 0.8)$ independently. For each DGP, we consider four configurations of data with $T = 100, 200, 400$ and $N = 100, 200$. To see how the position of the structural change affects the performance of our estimators, we consider $\tau_0 = 0.25$ and 0.5.

7.2 ESTIMATORS AND RESULTS

The number of pseudo factors in the equivalent model is estimated using IC_{p1} in Bai and Ng (2002) for Setups 1 and 2. For Setup 3, it is estimated using IC_{p1} in case $a = 1$ and IC_{p3} in case $a = 0.2$ and 0.6. The maximum number of factors is $rmax = 12$. Since estimating the number of pseudo factors is the first step of our estimation procedure, the performance of $\tilde{r}$ will affect the performance of $\tilde{k}$, which will further affect the performance of $\tilde{r}_1$, $\tilde{r}_2$ and the estimated pre-break and post-break factor spaces, it is worth discussing the choice of criterion in estimating the number of pseudo factors. As can be seen in the equivalent representation, the pseudo factors

[1] $x_i(R_i^2) = \frac{1-\rho^2}{1-\alpha^2} \frac{R_i^2}{1-R_i^2}$

induced by structural change are not as strong as factors with stable loadings in the original model[2] because a portion of their elements are zeros and the magnitude of those nonzero elements is small if the magnitude of structural change is small. Consequently, estimators of the number of factors which perform well in the normal case tend to underestimate the number of pseudo factors while estimators which tend to overestimate in the normal case perform well in estimating the number of pseudo factors. Moreover, the magnitudes of pseudo factors induced by structural change are not only absolutely smaller, but also relatively smaller, especially when the change point is not close to the middle of the sample. This decreases the applicability of the ER and GR estimators in Ahn and Horenstein (2013), whose performance rely on the factors being of similar magnitude. In our current setup, we found that among IC_{p1}, IC_{p2} in Bai and Ng (2002) and ER, GR in Ahn and Horenstein (2013), On the whole IC_{p1} performs best. Compared to IC_{p3}, IC_{p1} is more robust to serial correlation and heteroscedasticity of the errors, but IC_{p3} has an advantage in case the change point is far from middle or the magnitude of change is medium or small[3]. Since IC_{p1} and IC_{p3} are relatively less conservative, these findings are consistent with the above observations. In addition, we also found that underestimation of the number of pseudo factors deteriorates the performance of $\tilde{k}$ significantly more than overestimation. This is because $\tilde{k}$ is based on the second moment matrix of the estimated pseudo factors, hence underestimation will result in loss of information while overestimation will bring in extra noise. As long as the overestimation is not severe, these extra noise have very limited effect on performance of $\tilde{k}$. In view of all the above, we recommend choosing a less conservative criterion in estimating the number of pseudo factors.

The change point is estimated as in equation (11). We restrict $\tilde{k}$ to be in $[r_1, T-r_2]$ to avoid the singular matrix in subsequent estimation of the number of pre-break and post-break factors. This will not significantly affect the distribution of $\tilde{k}$ since the probability that $\tilde{k}$ falls out of $[r_1, T-r_2]$ is extremely small. To save space, we display the distributions of $\tilde{k}$ mainly for $(N,T) = (100, 100)$. Of course, the performance of $\tilde{k}$ improves as (N,T) increases. Figure 1 is the histogram of $\tilde{k}$ of Setup 1 for $(N,T) = (100, 100)$. Figures 2 and 3 are histograms of $\tilde{k}$ of Setup 2 for $(N,T) = (100, 100)$ and $(100, 200)$ respectively. Figures 4-6 are histograms of $\tilde{k}$ of Setup 3 for $(N,T) = (100, 100)$ with $a = 1, 0.6$ and 0.2, respectively. Each figure contains four subfigures corresponding to $\tau_0 = 0.25$ and 0.5 for $(\rho, \alpha, \beta) = (0, 0, 0)$ with homogeneous R^2 and $(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$ with heterogeneous

[2] All factors in the equivalent model are called pseudo factors, but not all pseudo factors are induced by structural change. Factors with stable loadings in the original model are still present in the equivalent model.

[3] Our comparison here is tentative, a comprehensive comparison in case the change point is far from middle or the magnitude of structural change is medium or small is left for a future study.

R^2 except for Figure 2, which contains two extra figures corresponding to $\tau_0 = 0.25$ and 0.5 for $(\rho, \alpha, \beta) = (0, 0.2, 0.2)$ with heterogeneous R^2. Under each subfigure, we also report the average and standard deviation of $\tilde{r}$ used in obtaining $\tilde{k}$. The number of replications is 1,000.

It is easy to see that in each subfigure the mass is concentrated in a small neighborhood of k_0. In most cases, the frequency that $\tilde{k}$ falls into $(k_0 - 8, k_0 + 8)$ is around 90%. This confirms our theoretical result, $\tilde{k} - k_0 = O_p(1)$. The worst case is Setup 2, which is also consistent with Theorem 2 since in this setup the variances of $\sum_{t=k_0+l}^{k_0-1}[vec(\Sigma_2 - \Sigma_1)]' y_t$ and $\sum_{t=k_0+1}^{k_0+l}[vec(\Sigma_2 - \Sigma_1)]' y_t$ are relatively larger. Nevertheless, Figure 3 shows that when T is increased to 200 the performance is much better. For Setups 1 and 3, $\tilde{k}$ performs quite well, especially for Setup 3 in which even with a decreased to 0.2, the performance deteriorates very little.

Comparing the left column with the right column of each figure, we can see that the performance of $\tilde{k}$ deteriorates as τ_0 moves from 0.5 to 0.25. This phenomenon can be attributed to two different reasons. First, as τ_0 gets closer to the boundary, the whole model behaves more similarly to no structural change. In such case the estimated change point will converge to the boundary. This is the main reason for Figures 2 and 3. Second, when τ_0 is close to the boundary, some pseudo factors in the equivalent model are weak and hence the PC estimator of these factors is noisy. In Setup 3, based on Theorem 2 and the fact that all factors and loadings are generated independently, it is not difficult to see that these weak factors are in $W(l)$ for $l = -1, -2, ...$, hence $\tilde{k} - k_0$ is likely to be negative. This is the main reason for Figures 4-6 and also explains the asymmetry of Figures 4-6. For Figure 1, both reasons are responsible since Setup 1 is a combination of Setups 2 and 3. In addition, as explained in Remark 2, the one-sided feature of Figures 2-3 is also due to the fact that all factors and loadings are generated independently. Since in reality Σ_G or Σ_Γ are unlikely to be diagonal, the distribution of $\tilde{k}$ will not be one-sided.

Comparing the first row with the second row of each figure, we can see that the performance of $\tilde{k}$ deteriorates for $(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$ with heterogeneous R^2. This is consistent with Theorem 2, since y_t is serial correlated when factors are serial correlated and serial correlation increases the variance of $\sum_{t=k_0+l}^{k_0-1}[vec(\Sigma_2 - \Sigma_1)]' y_t$ and $\sum_{t=k_0+1}^{k_0+l}[vec(\Sigma_2 - \Sigma_1)]' y_t$ for each l. Comparing the third row with the second and the first row of Figure 2, we can see that the deterioration is mainly due to serial correlation of the factors while the effect of serial correlation and cross-sectional dependence of idiosyncratic errors and heterogeneous R^2 is almost ignorable.

Based on $\tilde{k}$, we then split the sample and estimate the number of pre-break and post-break factors using IC_{p2} in Bai and Ng (2002) and GR in Ahn and Horenstein (2013), with maxima

$rmax_1 = 10$ and $rmax_2 = 10$. The performance of ER is similar and will not be reported. Based on $\tilde{k}$, $\tilde{r}_1$ and $\tilde{r}_2$, we then estimate the pre-break and post-break factors using the principal component method. To evaluate the performance, we calculate the R^2 of the multivariate regression of $\hat{F}_1^{\tilde{r}_1}(\tilde{k})$ on $F_1(\tilde{k})$ and $\hat{F}_2^{\tilde{r}_2}(\tilde{k})$ on $F_2(\tilde{k})$,

$$R^2_{\hat{F},F} = \frac{\left\| P_{F_1(\tilde{k})} \hat{F}_1^{\tilde{r}_1}(\tilde{k}) \right\|^2 + \left\| P_{F_2(\tilde{k})} \hat{F}_2^{\tilde{r}_2}(\tilde{k}) \right\|^2}{\left\| \hat{F}_1^{\tilde{r}_1}(\tilde{k}) \right\|^2 + \left\| \hat{F}_2^{\tilde{r}_2}(\tilde{k}) \right\|^2}.$$

Theorem 4 states that $R^2_{\hat{F},F}$ should be close to one if N and T are large.

Tables 1-3 report the percentage of underestimation and overestimation of $\tilde{r}_1$, $\tilde{r}_2$ and averages of $R^2_{\hat{F},F}$ over 1,000 replications. x/y denotes that the frequency of underestimation and overestimation is $x\%$ and $y\%$ respectively. On the whole, the performance of IC_{p2} and GR are similar. If we choose the better one in each case, the performance of $\tilde{r}_1$ and $\tilde{r}_2$ are quite good and in most cases close to the their correspondents based on the true change point k_0. For Setups 1 and 3, $(N, T) = (100, 200)$ is large enough to guarantee good performance in all cases while in case $\tau_0 = 0.5$, $(N, T) = (100, 100)$ is large enough. Note that for Setup 3, even with small magnitude of change $a = 0.2$, $\tilde{r}_1$ and $\tilde{r}_2$ still perform well. For Setup 2, $(N, T) = (100, 200)$ is large enough in all cases, except for the case with $\rho = 0.5$. The performance of $R^2_{\hat{F},F}$ is always good.

Comparing the results of $\tau_0 = 0.5$ with $\tau_0 = 0.25$ and $\rho = 0$ with $\rho = 0.5$ in each table, we can see that the deterioration pattern is in accord with that of $\tilde{k}$. This is not surprising since in the current setup, estimation error in $\tilde{k}$ is the main cause of misestimating $\tilde{r}_1$ and $\tilde{r}_2$. For $\tilde{r}_1$, underestimation of k_0 decreases the size of the pre-break subsample while overestimation increases the tendency of overestimating r_1. Comparing Tables 2 and 3, we can see that underestimation is less harmful. Finally, it is worth noting that there is still room for improvement of finite sample performance of $\tilde{r}_1$, $\tilde{r}_2$, either through improving the performance of $\tilde{k}$ or through choosing an estimator more robust to misspecification of change point among all estimators of the number of factors in the literature.

8 CONCLUSIONS

This paper studied the identification and estimation of a large dimensional factor model with a single large structural change. Both factor loadings and number of factors are allowed to be unstable. We proposed a least squares estimator of the change point and established its rate of convergence and limiting distribution. The main appeal of this estimator is that it does not require prior information of the number of factors and observability of the factors and it allows for a change in the number of factors. Based on this change point estimator, we are able to dissect the model into two separate stable models and consistently estimate the number of factors and the factor space. Our results are useful for analyzing a factor based model when structural change is present, for example, macroeconomic forecasting. It is widely acknowledged that forecasting performance breakdowns in the presence of large structural instability. Our results provide a promising solution for this problem. Our results also shed light on determining the number of dynamic factors when there exists structural instability. In addition, following the methods in Bai and Perron (1998), it will be straightforward to extend our results to the case with multiple changes.

A natural next step is to derive the limiting distribution of the estimated factors, factor loadings and common components as in Bai (2003). Many other issues are also on the agenda. For example, what are the asymptotic properties of the estimated change point, estimated number of factors and estimated factors when the factor process is I(1)? It will also be rewarding to further improve the finite sample performance of our change point estimator.

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APPENDIX

By symmetry, it suffices to study the case $k \le k_0$.

Define V_{NT} as the diagonal matrix of the first $r + q_1$ largest eigenvalues of $\frac{1}{NT}XX'$ in decreasing order and $\tilde{G}$ as $\sqrt{T}$times the corresponding eigenvector matrix, V as the diagonal matrix of eigenvalues of $\Sigma_\Gamma^{\frac{1}{2}}\Sigma_G\Sigma_\Gamma^{\frac{1}{2}}$ and Φ as the corresponding eigenvector matrix, $J = \frac{\Gamma'\Gamma}{N}\frac{G'\tilde{G}}{T}V_{NT}^{-1}$, $J_0 = \Sigma_\Gamma^{\frac{1}{2}}\Phi V^{-\frac{1}{2}}$. By definition, $\frac{1}{NT}XX'\tilde{G}V_{NT}^{-1} = \tilde{G}$. Plug in $X = G\Gamma' + E$, we have $\tilde{G} - GJ = \frac{1}{NT}(G\Gamma'E'\tilde{G} + E\Gamma G'\tilde{G} + EE'\tilde{G})V_{NT}^{-1}$ and

$$\tilde{g}_t - J'g_t = V_{NT}^{-1}(\frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\gamma_N(s,t) + \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\zeta_{st} + \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\eta_{st} + \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\xi_{st}), \quad (15)$$

where $\zeta_{st} = \frac{e_s'e_t}{N} - \gamma_N(s,t)$, $\eta_{st} = \frac{g_s'\Gamma'e_t}{N}$and $\xi_{st} = \frac{g_t'\Gamma'e_s}{N}$.

Next, define

$$\begin{aligned} z_t &= vec(\tilde{g}_t\tilde{g}_t' - J_0'g_tg_t'J_0) \\ &= vec[(\tilde{g}_t - J'g_t)(\tilde{g}_t - J'g_t)'] + vec[(\tilde{g}_t - J'g_t)g_t'J] + vec[J'g_t(\tilde{g}_t - J'g_t)'] \\ &\quad + vec[(J' - J_0')g_tg_t'(J' - J_0')'] + vec[(J' - J_0')g_tg_t'J_0] + vec[J_0'g_tg_t'(J' - J_0')']. \quad (16) \end{aligned}$$

It follows that

$$\begin{aligned} vec(\tilde{g}_t\tilde{g}_t') &= vec(\Sigma_1) + y_t + z_t \text{ for } t \le k_0, \\ vec(\tilde{g}_t\tilde{g}_t') &= vec(\Sigma_2) + y_t + z_t \text{ for } t > k_0, \quad (17) \end{aligned}$$

where Σ_1, Σ_2 and y_t are defined in Section 4.

For $k \le k_0$,

$$vec(\tilde{\Sigma}_1) = vec(\Sigma_1) + \frac{1}{k}\sum\nolimits_{t=1}^{k} y_t + \frac{1}{k}\sum\nolimits_{t=1}^{k} z_t, \quad (18)$$

$$\begin{aligned} vec(\tilde{\Sigma}_2) &= vec(\Sigma_1) + \frac{T-k_0}{T-k}[vec(\Sigma_2) - vec(\Sigma_1)] + \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} y_t + \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} z_t \\ &= \frac{k_0-k}{T-k}[vec(\Sigma_1) - vec(\Sigma_2)] + vec(\Sigma_2) + \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} y_t + \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} z_t (19) \end{aligned}$$

Define

$$a_k = \frac{T-k_0}{T-k}[vec(\Sigma_2) - vec(\Sigma_1)], b_k = \frac{k_0-k}{T-k}[vec(\Sigma_1) - vec(\Sigma_2)], \quad (20)$$

$$\bar{y}_1 = \frac{1}{k}\sum\nolimits_{t=1}^{k} y_t,\ \bar{y}_2 = \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} y_t, \quad (21)$$

$$\bar{z}_1 = \frac{1}{k}\sum\nolimits_{t=1}^{k} z_t,\ \bar{z}_2 = \frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} z_t. \quad (22)$$

It follows that

$$\begin{aligned} vec(\tilde{\Sigma}_1) &= vec(\Sigma_1) + \bar{y}_1 + \bar{z}_1, \\ vec(\tilde{\Sigma}_2) &= vec(\Sigma_1) + a_k + \bar{y}_2 + \bar{z}_2 = vec(\Sigma_2) + b_k + \bar{y}_2 + \bar{z}_2, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \tilde{S}(k) &= \sum_{t=1}^{k}(y_t + z_t - \bar{y}_1 - \bar{z}_1)'(y_t + z_t - \bar{y}_1 - \bar{z}_1) \\ &\quad + \sum_{t=k+1}^{k_0}(y_t + z_t - \bar{y}_2 - \bar{z}_2 - a_k)'(y_t + z_t - \bar{y}_2 - \bar{z}_2 - a_k) \\ &\quad + \sum_{t=k_0+1}^{T}(y_t + z_t - \bar{y}_2 - \bar{z}_2 - b_k)'(y_t + z_t - \bar{y}_2 - \bar{z}_2 - b_k) \\ &= \sum_{t=1}^{k}(y_t + z_t - \bar{y}_1 - \bar{z}_1)'(y_t + z_t - \bar{y}_1 - \bar{z}_1) \\ &\quad + \sum_{t=k+1}^{T}(y_t + z_t - \bar{y}_2 - \bar{z}_2)'(y_t + z_t - \bar{y}_2 - \bar{z}_2) \\ &\quad + (k_0 - k)a_k' a_k + (T - k_0)b_k' b_k \\ &\quad - 2a_k' \sum_{t=k+1}^{k_0}(y_t + z_t - \bar{y}_2 - \bar{z}_2) - 2b_k' \sum_{t=k_0+1}^{T}(y_t + z_t - \bar{y}_2 - \bar{z}_2) \\ &= (k_0 - k)a_k' a_k + (T - k_0)b_k' b_k + \sum_{t=1}^{T}(y_t + z_t)'(y_t + z_t) \\ &\quad - k(\bar{y}_1 + \bar{z}_1)'(\bar{y}_1 + \bar{z}_1) - (T - k)(\bar{y}_2 + \bar{z}_2)'(\bar{y}_2 + \bar{z}_2) \\ &\quad - 2a_k' \sum_{t=k+1}^{k_0}(y_t + z_t - \bar{y}_2 - \bar{z}_2) - 2b_k' \sum_{t=k_0+1}^{T}(y_t + z_t - \bar{y}_2 - \bar{z}_2), \end{aligned} \tag{24}$$

$$\begin{aligned} \tilde{S}(k) - \tilde{S}(k_0) &= (k_0 - k)a_k' a_k + (T - k_0)b_k' b_k \\ &\quad - \{\frac{1}{k}[\sum_{t=1}^{k}(y_t + z_t)]'[\sum_{t=1}^{k}(y_t + z_t)] - \frac{1}{k_0}[\sum_{t=1}^{k_0}(y_t + z_t)]'[\sum_{t=1}^{k_0}(y_t + z_t)]\} \\ &\quad - \{\frac{1}{T - k}[\sum_{t=k+1}^{T}(y_t + z_t)]'[\sum_{t=k+1}^{T}(y_t + z_t)] \\ &\quad - \frac{1}{T - k_0}[\sum_{t=k_0+1}^{T}(y_t + z_t)]'[\sum_{t=k_0+1}^{T}(y_t + z_t)]\} \\ &\quad - 2a_k' \sum_{t=k+1}^{k_0}(y_t + z_t) - 2b_k' \sum_{t=k_0+1}^{T}(y_t + z_t) \\ &\quad + 2[(k_0 - k)a_k + (T - k_0)b_k]'(\bar{y}_2 + \bar{z}_2) \\ &= A + B + C + D + E + F + G. \end{aligned} \tag{25}$$

A PROOF OF PROPOSITION 2

Proof. To show $\tilde{\tau} - \tau_0 = o_p(1)$, we need to show for any $\epsilon > 0$ and any $\eta > 0$, $P(|\tilde{\tau} - \tau_0| > \eta) < \epsilon$ as $(N, T) \to \infty$. For the given η, define $D = \{k : (\tau_0 - \eta)T \leq k \leq (\tau_0 + \eta)T\}$, we need to show $P(\tilde{k} \in D^c) < \epsilon$.

$\tilde{k} = \arg\min \tilde{S}(k)$, hence $\tilde{S}(\tilde{k}) - \tilde{S}(k_0) \leq 0$. If $\tilde{k} \in D^c$, then $\min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0$. This implies $P(\tilde{k} \in D^c) \leq P(\min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0)$, hence it suffices to show for any given $\epsilon > 0$ and $\eta > 0$, $P(\min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0) < \epsilon$ as $(N, T) \to \infty$.

If $\omega \in \{\omega : \min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0\}$ and $\arg\min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) = k^*$, then $\tilde{S}(k^*) - \tilde{S}(k_0) \leq 0$, hence $\frac{\tilde{S}(k^*) - \tilde{S}(k_0)}{|k^* - k_0|} \leq 0$ and it follows $\min_{k \in D^c} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq \frac{\tilde{S}(k^*) - \tilde{S}(k_0)}{|k^* - k_0|} \leq 0$. Hence $\omega \in \{\omega : \min_{k \in D^c} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq 0\}$, this implies $\{\omega : \min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0\} \subseteq \{\omega : \min_{k \in D^c} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq 0\}$. Similarly, $\{\omega : \min_{k \in D^c} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq 0\} \subseteq \{\omega : \min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0\}$. Therefore, $\{\omega : \min_{k \in D^c} \tilde{S}(k) - \tilde{S}(k_0) \leq 0\} = \{\omega : \min_{k \in D^c} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq 0\}$.

By symmetry, it suffices to study the case $k < k_0$.

$$
\begin{aligned}
P(\min_{k \in D^c, k < k_0} \tilde{S}(k) - \tilde{S}(k_0) \leq 0) &= P(\min_{k \in D^c, k < k_0} \frac{\tilde{S}(k) - \tilde{S}(k_0)}{|k - k_0|} \leq 0) \\
&\leq P(\min_{k \in D^c, k < k_0} \frac{A + B}{|k - k_0|} \leq \sup_{k \in D^c, k < k_0} \frac{|C|}{|k_0 - k|} + \sup_{k \in D^c, k < k_0} \frac{|D|}{|k_0 - k|} \\
&\quad + \sup_{k \in D^c, k < k_0} \frac{|E|}{|k_0 - k|} + \sup_{k \in D^c, k < k_0} \frac{|F|}{|k_0 - k|} + \sup_{k \in D^c, k < k_0} \frac{|G|}{|k_0 - k|}).
\end{aligned}
$$

We will show the right hand side are dominated by the left hand side.

First consider term $A + B$,

$$
\begin{aligned}
\min_{k \in D^c, k < k_0} \frac{A + B}{|k - k_0|} &\geq \min_{k \in D^c, k < k_0} \frac{A}{|k_0 - k|} = \min_{k \in D^c, k < k_0} a_k' a_k \\
&= \min_{k \in D^c, k < k_0} (\frac{T - k_0}{T - k})^2 [vec(\Sigma_2 - \Sigma_1)]' [vec(\Sigma_2 - \Sigma_1)] \\
&\geq (1 - \tau_0)^2 \left\| \Sigma_2 - \Sigma_1 \right\|^2 = (1 - \tau_0)^2 \left\| J_0 \right\|^4 \left\| \Sigma_{G,2} - \Sigma_{G,1} \right\|^2.
\end{aligned}
$$

Next consider term C,

$$
\begin{aligned}
C &= -\{\frac{1}{k}[\sum_{t=1}^{k} (y_t + z_t)]' [\sum_{t=1}^{k} (y_t + z_t)] - \frac{1}{k_0}[\sum_{t=1}^{k_0} (y_t + z_t)]' [\sum_{t=1}^{k_0} (y_t + z_t)]\} \\
&= -\frac{k_0 - k}{k_0} \frac{1}{k}[\sum_{t=1}^{k} (y_t + z_t)]' [\sum_{t=1}^{k} (y_t + z_t)] + 2\frac{1}{k_0}[\sum_{t=1}^{k} (y_t + z_t)]' [\sum_{t=k+1}^{k_0} (y_t + z_t)] \\
&\quad + \frac{k_0 - k}{k_0} \frac{1}{k_0 - k}[\sum_{t=k+1}^{k_0} (y_t + z_t)]' [\sum_{t=k+1}^{k_0} (y_t + z_t)]. \qquad (26)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\left| \frac{C}{k_0 - k} \right| &\leq \left| \frac{1}{k_0} \frac{1}{k}[\sum_{t=1}^{k} (y_t + z_t)]' [\sum_{t=1}^{k} (y_t + z_t)] \right| \\
&\quad + \left| 2\frac{1}{k_0} \frac{1}{k_0 - k}[\sum_{t=1}^{k} (y_t + z_t)]' [\sum_{t=k+1}^{k_0} (y_t + z_t)] \right| \\
&\quad + \left| \frac{1}{k_0} \frac{1}{k_0 - k}[\sum_{t=k+1}^{k_0} (y_t + z_t)]' [\sum_{t=k+1}^{k_0} (y_t + z_t)] \right| \\
&= C_1 + C_2 + C_3.
\end{aligned}
$$

For C_1,

$$\begin{aligned}\sup_{k\in D^c,k<k_0} C_1 &= \sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k}\left\|\sum\nolimits_{t=1}^{k}(y_t+z_t)\right\|^2 \\ &\leq \sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k}(\left\|\sum\nolimits_{t=1}^{k}y_t\right\|^2+\left\|\sum\nolimits_{t=1}^{k}z_t\right\|^2+2\left\|\sum\nolimits_{t=1}^{k}y_t\right\|\left\|\sum\nolimits_{t=1}^{k}z_t\right\|) \\ &\leq 2\sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k}\left\|\sum\nolimits_{t=1}^{k}y_t\right\|^2+2\sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k}\left\|\sum\nolimits_{t=1}^{k}z_t\right\|^2.\end{aligned}$$

By Hajek-Renyi inequality, $\sup_{k\in D^c,k<k_0}\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}y_t\right\|=O_p(\sqrt{\log T})$, hence the first term is $O_p(\frac{\log T}{T})$. By part (1) of Lemma 7, the second term is $o_p(1)$, hence $\sup_{k\in D^c,k<k_0} C_1=o_p(1)$.

For C_2,

$$\begin{aligned}\sup_{k\in D^c,k<k_0} C_2 &\leq 2\sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=1}^{k}(y_t+z_t)\right\|\left\|\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)\right\| \\ &\leq 2\sup_{k\in D^c,k<k_0}(\left\|\frac{1}{k_0}\sum\nolimits_{t=1}^{k}y_t\right\|+\left\|\frac{1}{k_0}\sum\nolimits_{t=1}^{k}z_t\right\|)(\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}y_t\right\| \\ &\quad+\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}z_t\right\|) \\ &\leq 2(\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0}\sum\nolimits_{t=1}^{k}y_t\right\|+\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0}\sum\nolimits_{t=1}^{k}z_t\right\|) \\ &\quad(\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}y_t\right\|+\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}z_t\right\|).\end{aligned}$$

By Hajek-Renyi inequality, $\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0}\sum_{t=1}^{k}y_t\right\|=O_p(\frac{1}{\sqrt{T}})$, $\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}y_t\right\|=O_p(\frac{1}{\sqrt{T}})$. By parts (3) and (5) of Lemma 7, $\sup_{k\in D^c,k<k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|=o_p(1)$, $\sup_{k\in D^c,k<k_0}\frac{1}{|k_0-k|}\left\|\sum_{t=k+1}^{k_0}z_t\right\|=o_p(1)$, hence $\sup_{k\in D^c,k<k_0} C_2=[O_p(\frac{1}{\sqrt{T}})+o_p(1))][O_p(\frac{1}{\sqrt{T}})+o_p(1)]=o_p(1)$.

For C_3,

$$\begin{aligned}\sup_{k\in D^c,k<k_0} C_3 &= \sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)\right\|^2 \\ &\leq \sup_{k\in D^c,k<k_0} \frac{1}{k_0}\frac{1}{k_0-k}(\left\|\sum\nolimits_{t=k+1}^{k_0}y_t\right\|+\left\|\sum\nolimits_{t=k+1}^{k_0}z_t\right\|)^2 \\ &\leq 2\frac{1}{k_0}\sup_{k\in D^c,k<k_0}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=k+1}^{k_0}y_t\right\|^2+2\frac{1}{k_0}\sup_{k\in D^c,k<k_0}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=k+1}^{k_0}z_t\right\|^2.\end{aligned}$$

By Hajek-Renyi inequality, $\sup_{k\in D^c,k<k_0}\left\|\frac{1}{\sqrt{k_0-k}}\sum_{t=k+1}^{k_0}y_t\right\|=O_p(1)$, hence the first term is $O_p(\frac{1}{T})$. By part (7) of Lemma 7, $\sup_{k\in D^c,k<k_0}\frac{1}{k_0}\frac{1}{|k_0-k|}\left\|\sum_{t=k+1}^{k_0}z_t\right\|^2=o_p(1)$, hence $\sup_{k\in D^c,k<k_0} C_3=o_p(1)$.

Therefore, $\sup_{k\in D^c,k<k_0}\left|\frac{C}{k_0-k}\right|\leq\sup_{k\in D^c,k<k_0} C_1+\sup_{k\in D^c,k<k_0} C_2+\sup_{k\in D^c,k<k_0} C_3=o_p(1)$.

Similarly,

$$\begin{aligned}\left|\frac{D}{k_0-k}\right| &\le \left|\frac{1}{T-k_0}\frac{1}{T-k}[\sum\nolimits_{t=k_0+1}^{T}(y_t+z_t)]'[\sum\nolimits_{t=k_0+1}^{T}(y_t+z_t)]\right| \\ &\quad+\left|2\frac{1}{T-k}\frac{1}{k_0-k}[\sum\nolimits_{t=k_0+1}^{T}(y_t+z_t)]'[\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)]\right| \\ &\quad+\left|\frac{1}{T-k}\frac{1}{k_0-k}[\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)]'[\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)]\right| \\ &= D_1+D_2+D_3. \end{aligned} \tag{27}$$

$$\begin{aligned}\sup_{k\in D^c,k<k_0} D_1 &\le 2\sup_{k\in D^c,k<k_0}\frac{1}{T-k_0}\frac{1}{T-k}\left\|\sum\nolimits_{t=k_0+1}^{T}y_t\right\|^2+2\sup_{k\in D^c,k<k_0}\frac{1}{T-k_0}\frac{1}{T-k}\left\|\sum\nolimits_{t=k_0+1}^{T}z_t\right\|^2 \\ &= O_p(\frac{1}{T})+o_p(1)=o_p(1),\end{aligned}$$

where the equality follows from Hajek-Renyi inequality and part (9) of Lemma 7.

$$\begin{aligned}\sup_{k\in D^c,k<k_0} D_2 &\le 2(\sup_{k\in D^c,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k_0+1}^{T}y_t\right\|+\sup_{k\in D^c,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k_0+1}^{T}z_t\right\|) \\ &\quad(\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}y_t\right\|+\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}z_t\right\|) \\ &= (O_p(\frac{1}{\sqrt{T}})+o_p(1))(O_p(\frac{1}{\sqrt{T}})+o_p(1))=o_p(1),\end{aligned}$$

where the equality follows from Hajek-Renyi inequality and parts (9) and (5) of Lemma 7.

$$\begin{aligned}\sup_{k\in D^c,k<k_0} D_3 &\le 2\sup_{k\in D^c,k<k_0}\frac{1}{T-k}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=k+1}^{k_0}y_t\right\|^2+2\sup_{k\in D^c,k<k_0}\frac{1}{T-k}\frac{1}{k_0-k}\left\|\sum\nolimits_{t=k+1}^{k_0}z_t\right\|^2 \\ &= O_p(\frac{1}{T})+o_p(1)=o_p(1),\end{aligned}$$

where the equality follows from Hajek-Renyi inequality and part (7) of Lemma 7.

Therefore, $\sup_{k\in D^c,k<k_0}\left|\frac{D}{k_0-k}\right|\le\sup_{k\in D^c,k<k_0}D_1+\sup_{k\in D^c,k<k_0}D_2+\sup_{k\in D^c,k<k_0}D_3=o_p(1)$.

Next consider term E.

$$\begin{aligned}\sup_{k\in D^c,k<k_0}\left|\frac{E}{k_0-k}\right| &= 2\sup_{k\in D^c,k<k_0}\frac{1}{k_0-k}\left|a_k'\sum\nolimits_{t=k+1}^{k_0}(y_t+z_t)\right| \\ &\le 2\|\Sigma_2-\Sigma_1\|\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}y_t\right\| \\ &\quad+2\|\Sigma_2-\Sigma_1\|\sup_{k\in D^c,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}z_t\right\|.\end{aligned} \tag{28}$$

By Hajek-Renyi inequality, the first term is $O_p(\frac{1}{\sqrt{T}})$. By part (5) of Lemma 7, the second term is $o_p(1)$. Therefore, $\sup_{k\in D^c,k<k_0}\left|\frac{E}{k_0-k}\right|=o_p(1)$.

For term F,

$$\sup_{k\in D^c,k<k_0}\left|\frac{F}{k_0-k}\right| \leq 2\sup_{k\in D^c,k<k_0}\frac{\|b_k\|\left\|\sum_{t=k_0+1}^{T}(y_t+z_t)\right\|}{|k_0-k|} \leq 2\|\Sigma_1-\Sigma_2\|\left\|\frac{1}{T-k_0}\sum\nolimits_{t=k_0+1}^{T}y_t\right\| +2\|\Sigma_1-\Sigma_2\|\left\|\frac{1}{T-k_0}\sum\nolimits_{t=k_0+1}^{T}z_t\right\|. \quad (29)$$

By Hajek-Renyi inequality, the first term is $O_p(\frac{1}{\sqrt{T}})$. By part (9) of Lemma 7, $\left\|\frac{1}{T-k_0}\sum_{t=k_0+1}^{T}z_t\right\| \leq \sup_{k\leq k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T}z_t\right\| = o_p(1)$. Therefore, $\sup_{k\in D^c,k<k_0}\left|\frac{F}{k_0-k}\right| = o_p(1)$.

For term G, note that $(k_0-k)a_k = (T-k_0)b_k$,

$$\begin{aligned}\sup_{k\in D^c,k<k_0}\left|\frac{G}{k_0-k}\right| &= 4\sup_{k\in D^c,k<k_0}\left|a_k'(\bar{y}_2+\bar{z}_2)\right| \\ &\leq 4\sup_{k\in D^c,k<k_0}\frac{T-k_0}{T-k}\|\Sigma_2-\Sigma_1\|\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}(y_t+z_t)\right\| \\ &\leq 4\|\Sigma_2-\Sigma_1\|\sup_{k\in D^c,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}y_t\right\| \\ &\quad +4\|\Sigma_2-\Sigma_1\|\sup_{k\in D^c,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}z_t\right\|. \quad (30)\end{aligned}$$

$\sup_{k\in D^c,k<k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T}y_t\right\| \leq \sup_{k<k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T}y_t\right\| \leq \frac{1}{1-\tau_0}(\sup_{k<k_0}\frac{1}{T}\left\|\sum_{t=k+1}^{k_0}y_t\right\|+\sup_{k>k_0}\frac{1}{T}\left\|\sum_{t=k_0+1}^{k}y_t\right\|)$, hence by Hajek-Renyi inequality the first term is $O_p(\frac{1}{\sqrt{T}})$. By part (9) of Lemma 7, the second term is $o_p(1)$. Therefore, $\sup_{k\in D^c,k<k_0}\left|\frac{G}{k_0-k}\right| = o_p(1)$. ■

B PROOF OF THEOREM 1

Proof. To show $\tilde{k}-k_0 = O_p(1)$, we need to show for any $\epsilon > 0$ there exist $M > 0$ such that $P(\left|\tilde{k}-k_0\right| > M) < \epsilon$ as $(N,T)\to\infty$. By Proposition 2, for any $\epsilon > 0$ and $\min\{\tau_0, 1-\tau_0\} > \eta > 0$, $P(\tilde{k}\in D^c) < \epsilon$ as $(N,T)\to\infty$. For the given η and M, define $D_M = \{k : (\tau_0-\eta)T \leq k \leq (\tau_0+\eta)T$, $|k-k_0| > M\}$, then $P(\left|\tilde{k}-k_0\right| > M) = P(\tilde{k}\in D^c) + P(\tilde{k}\in D_M)$. Hence it suffices to show that for any $\epsilon > 0$ and $\eta > 0$, there exist $M > 0$ such that $P(\tilde{k}\in D_M) < \epsilon$ as $(N,T)\to\infty$. Again by symmetry, it suffices to study the case $k < k_0$. Similar to the proof of Proposition 2, it suffices to show for any given $\epsilon > 0$ and $\eta > 0$, there exist $M > 0$ such that $P(\min_{k\in D_M,k<k_0}\frac{A+B}{|k_0-k|} \leq \sup_{k\in D_M,k<k_0}\left|\frac{C}{k_0-k}\right| + \sup_{k\in D_M,k<k_0}\left|\frac{D}{k_0-k}\right| + \sup_{k\in D_M,k<k_0}\left|\frac{E}{k_0-k}\right| + \sup_{k\in D_M,k<k_0}\left|\frac{F}{k_0-k}\right| + \sup_{k\in D_M,k<k_0}\left|\frac{G}{k_0-k}\right|) < \epsilon$ as $(N,T)\to\infty$.

First consider term $A+B$,

$$\begin{aligned}\min_{k\in D_M,k<k_0}\frac{A+B}{|k_0-k|} &= \min_{k\in D_M,k<k_0} a_k'a_k=\min_{k\in D_M,k<k_0}(\frac{T-k_0}{T-k})^2[vec(\Sigma_2-\Sigma_1)]'[vec(\Sigma_2-\Sigma_1)]\\ &\geq (1-\tau_0)^2\left\|\Sigma_2-\Sigma_1\right\|^2=(1-\tau_0)^2\left\|J_0\right\|^4\left\|\Sigma_{G,2}-\Sigma_{G,1}\right\|^2.\end{aligned}$$

Next consider term C. Similar to the proof of Proposition 2,

$$\sup_{k\in D_M,k<k_0}\left|\frac{C}{k_0-k}\right|\leq\sup_{k\in D,k<k_0}\left|\frac{C}{k_0-k}\right|\leq\sup_{k\in D,k<k_0}C_1+\sup_{k\in D,k<k_0}C_2+\sup_{k\in D,k<k_0}C_3.$$

For C_1,

$$\sup_{k\in D,k<k_0}C_1\leq 2\sup_{k\in D,k<k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}y_t\right\|^2+2\sup_{k\in D,k<k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}z_t\right\|^2.$$

By Hajek-Renyi inequality, $\sup_{k\in D,k<k_0}\left\|\frac{1}{\sqrt{k}}\sum_{t=1}^{k}y_t\right\|=O_p(1)$, hence the first term is $O_p(\frac{1}{T})$. By part (2) of Lemma 7, the second term is $o_p(1)$, hence $\sup_{k\in D,k<k_0}C_1=o_p(1)$.

For C_2,

$$\begin{aligned}\sup_{k\in D,k<k_0}C_2 &\leq 2(\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0}\sum_{t=1}^{k}y_t\right\|+\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0}\sum_{t=1}^{k}z_t\right\|)(\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}y_t\right\|\\ &\quad+\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}z_t\right\|).\end{aligned}$$

By Hajek-Renyi inequality, $\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0}\sum_{t=1}^{k}y_t\right\|=O_p(\frac{1}{\sqrt{T}})$, $\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0}y_t\right\|=O_p(1)$. By parts (4) and (6) of Lemma 7, $\sup_{k\in D,k<k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|=o_p(1)$, $\sup_{k\in D,k<k_0}\frac{1}{|k_0-k|}\left\|\sum_{t=k+1}^{k_0}z_t\right\|=o_p(1)$. Hence $\sup_{k\in D,k<k_0}C_2=[O_p(\frac{1}{\sqrt{T}})+o_p(1)][O_p(1)+o_p(1)]=o_p(1)$.

For C_3,

$$\sup_{k\in D,k<k_0}C_3\leq 2\frac{1}{k_0}\sup_{k\in D,k<k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}y_t\right\|^2+2\frac{1}{k_0}\sup_{k\in D,k<k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}z_t\right\|^2.$$

By Hajek-Renyi inequality, $\sup_{k\in D,k<k_0}\left\|\frac{1}{\sqrt{k_0-k}}\sum_{t=k+1}^{k_0}y_t\right\|=O_p(\sqrt{\log T})$, hence the first term is $O_p(\frac{\log T}{T})$. By part (8) of Lemma 7, $\sup_{k\in D,k<k_0}\frac{1}{k_0}\frac{1}{|k_0-k|}\left\|\sum_{t=k+1}^{k_0}z_t\right\|^2=o_p(1)$. Hence $\sup_{k\in D,k<k_0}C_3=o_p(1)$. Therefore, $\sup_{k\in D_M,k<k_0}\left|\frac{C}{k_0-k}\right|=o_p(1)$.

Similarly,

$$\sup_{k\in D_M,k<k_0}\left|\frac{D}{k_0-k}\right|\leq\sup_{k\in D,k<k_0}\left|\frac{D}{k_0-k}\right|\leq\sup_{k\in D,k<k_0}D_1+\sup_{k\in D,k<k_0}D_2+\sup_{k\in D,k<k_0}D_3.$$

$$\sup_{k\in D,k<k_0} D_1 \leq 2\sup_{k\in D,k<k_0}\frac{1}{T-k_0}\frac{1}{T-k}\left\|\sum_{t=k_0+1}^{T} y_t\right\|^2 + 2\sup_{k\in D,k<k_0}\frac{1}{T-k_0}\frac{1}{T-k}\left\|\sum_{t=k_0+1}^{T} z_t\right\|^2$$
$$= O_p(\frac{1}{T}) + o_p(1) = o_p(1),$$

where the equality follows from Hajek-Renyi inequality and part (9) of Lemma 7.

$$\sup_{k\in D,k<k_0} D_2 \leq 2(\sup_{k\in D,k<k_0}\left\|\frac{1}{T-k}\sum_{t=k_0+1}^{T} y_t\right\| + \sup_{k\in D,k<k_0}\left\|\frac{1}{T-k}\sum_{t=k_0+1}^{T} z_t\right\|)$$
$$(\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} y_t\right\| + \sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} z_t\right\|)$$
$$= (O_p(\frac{1}{\sqrt{T}}) + o_p(1))(O_p(1) + o_p(1)) = o_p(1),$$

where the equality follows from Hajek-Renyi inequality and parts (9) and (6) of Lemma 7.

$$\sup_{k\in D,k<k_0} D_3 \leq 2\sup_{k\in D,k<k_0}\frac{1}{T-k}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0} y_t\right\|^2 + 2\sup_{k\in D,k<k_0}\frac{1}{T-k}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0} z_t\right\|^2$$
$$= O_p(\frac{\log T}{T}) + o_p(1) = o_p(1),$$

where the equality follows from Hajek-Renyi inequality and part (8) of Lemma 7.

Therefore, $\sup_{k\in D_M,k<k_0}\left|\frac{D}{k_0-k}\right| = o_p(1)$.

Next consider term E.

$$\sup_{k\in D_M,k<k_0}\left|\frac{E}{k_0-k}\right| \leq 2\left\|\Sigma_2-\Sigma_1\right\|\sup_{k\in D_M,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} y_t\right\|$$
$$+2\left\|\Sigma_2-\Sigma_1\right\|\sup_{k\in D_M,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} z_t\right\|.$$

For any given $\delta > 0$, $P(2\left\|\Sigma_2-\Sigma_1\right\|\sup_{k\in D_M,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} y_t\right\| \geq \delta(1-\tau_0)^2\left\|\Sigma_2-\Sigma_1\right\|^2) = P(\sup_{k\in D_M,k<k_0}\left\|\frac{1}{k_0-k}\sum_{t=k+1}^{k_0} y_t\right\| \geq \delta\frac{(1-\tau_0)^2}{2}\left\|\Sigma_2-\Sigma_1\right\|) \leq \frac{C}{M\delta^2} \to 0$ as $M\to\infty$, hence the first term is dominated by $\min_{k\in D_M,k<k_0}\frac{A+B}{|k_0-k|}$. By part (6) of Lemma 7, the second term is $o_p(1)$. Therefore, $\sup_{k\in D_M,k<k_0}\left|\frac{E}{k_0-k}\right|$ is dominated by $\min_{k\in D_M,k<k_0}\frac{A+B}{|k_0-k|}$ as $M\to\infty$.

For term F,

$$\sup_{k\in D_M,k<k_0}\left|\frac{F}{k_0-k}\right| \leq \sup_{k\in D,k<k_0}\left|\frac{F}{k_0-k}\right| \leq 2\sup_{k\in D,k<k_0}\frac{\left\|b_k\right\|\left\|\sum_{t=k_0+1}^{T}(y_t+z_t)\right\|}{|k_0-k|}$$
$$\leq 2\left\|\Sigma_1-\Sigma_2\right\|\left\|\frac{1}{T-k_0}\sum_{t=k_0+1}^{T} y_t\right\| + 2\left\|\Sigma_1-\Sigma_2\right\|\left\|\frac{1}{T-k_0}\sum_{t=k_0+1}^{T} z_t\right\|.$$

By Hajek-Renyi inequality, the first term is $O_p(\frac{1}{\sqrt{T}})$. By part (9) of Lemma 7, the second term is $o_p(1)$. Therefore, $\sup_{k\in D_M,k<k_0}\left|\frac{F}{k_0-k}\right| = o_p(1)$.

For term G,

$$\sup_{k\in D_M,k<k_0}\left|\frac{G}{k_0-k}\right| \leq \sup_{k\in D,k<k_0}\left|\frac{G}{k_0-k}\right| \leq 4\left\|\Sigma_2-\Sigma_1\right\| \sup_{k\in D,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} y_t\right\| +4\left\|\Sigma_2-\Sigma_1\right\| \sup_{k\in D,k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T} z_t\right\|.$$

$\sup\limits_{k\in D,k<k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T} y_t\right\| \leq \sup\limits_{k<k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T} y_t\right\| \leq \frac{1}{1-\tau_0}(\sup\limits_{k<k_0}\frac{1}{T}\left\|\sum_{t=k+1}^{k_0} y_t\right\|+\sup\limits_{k>k_0}\frac{1}{T}\left\|\sum_{t=k_0+1}^{k} y_t\right\|)$, hence by Hajek-Renyi inequality, the first term is $O_p(\frac{1}{\sqrt{T}})$. By part (9) of Lemma 7, the second term is $o_p(1)$. Therefore, $\sup\limits_{k\in D_M,k<k_0}\left|\frac{G}{k_0-k}\right| = o_p(1)$. ■

C PROOF OF THEOREM 2

Proof. Define $V(k) = \tilde{S}(k) - \tilde{S}(k_0)$, $U(k) = A + E = (k_0-k)a_k'a_k - 2a_k'\sum_{t=k+1}^{k_0}(y_t+z_t)$ for $k<k_0$. For any fixed constant $M<\infty$, define $V^M(k) = V(k)$ for $|k_0-k|<M$, $U^M(k) = U(k)$ for $|k_0-k|<M$, $W^M(l) = W(l)$ for $|l|<M$. $V^M(k)$, $U^M(k)$ and $W^M(l)$ are all finite dimensional random vector.

Step 1: $V^M(k) \xrightarrow{p} U^M(k)$ as $(N,T)\to\infty$ for any fixed $M<\infty$.

By symmetry we only need to study the case $k<k_0$. It suffices to show $\sup\limits_{|k_0-k|<M,k<k_0}|V(k)-U(k)| = o_p(1)$.

$$\sup_{|k_0-k|<M,k<k_0}|V(k)-U(k)| \leq \sup_{|k_0-k|<M,k<k_0}|B| + \sup_{|k_0-k|<M,k<k_0}|C| + \sup_{|k_0-k|<M,k<k_0}|D| + \sup_{|k_0-k|<M,k<k_0}|F| + \sup_{|k_0-k|<M,k<k_0}|G|.$$

$$\sup_{|k_0-k|<M,k<k_0}|B| = \sup_{|k_0-k|<M,k<k_0}(T-k_0)(\frac{k_0-k}{T-k})^2\left\|\Sigma_2-\Sigma_1\right\|^2 = O(\frac{1}{T}) = o(1).$$

$$\sup_{|k_0-k|<M,k<k_0}|C| \leq M\sup_{k\in D,k<k_0}\left|\frac{C}{k_0-k}\right| = o_p(1).$$

Similarly, $\sup\limits_{|k_0-k|<M,k<k_0}|D|$, $\sup\limits_{|k_0-k|<M,k<k_0}|F|$ and $\sup\limits_{|k_0-k|<M,k<k_0}|G|$ are all $o_p(1)$.

Step 2: $U^M(k) \xrightarrow{d} W^M(k-k_0)$ as $(N,T)\to\infty$ for any fixed $M<\infty$.

$U^M(k) = (k_0-k)a_k'a_k - 2a_k'\sum_{t=k+1}^{k_0}(y_t+z_t)$, for $|k_0-k|<M$ and $k<k_0$.

For $|k_0-k|<M$,

$$\begin{aligned}(k_0-k)a_k'a_k &= (k_0-k)\left\|\Sigma_2-\Sigma_1\right\|^2 + (k_0-k)[(\frac{T-k_0}{T-k})^2-1]\left\|\Sigma_2-\Sigma_1\right\|^2 \\ &= (k_0-k)\left\|\Sigma_2-\Sigma_1\right\|^2 + O(\frac{1}{T}).\end{aligned}$$

By part (6) of Lemma 7,

$$\sup_{|k_0-k|<M,k<k_0}\left|-2a_k'\sum\nolimits_{t=k+1}^{k_0} z_t\right| \leq 2M\left\|\Sigma_2-\Sigma_1\right\|\sup_{|k_0-k|<M,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} z_t\right\|$$
$$\leq 2M\left\|\Sigma_2-\Sigma_1\right\|\sup_{k\in D,k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} z_t\right\| = o_p(1),$$

Next,

$$-2a_k'\sum\nolimits_{t=k+1}^{k_0} y_t = -2[vec(\Sigma_2-\Sigma_1)]'\sum\nolimits_{t=k+1}^{k_0} y_t - 2(\frac{T-k_0}{T-k}-1)[vec(\Sigma_2-\Sigma_1)]'\sum\nolimits_{t=k+1}^{k_0} y_t,$$

and

$$\sup_{|k_0-k|<M,k<k_0}\left|-2(\frac{T-k_0}{T-k}-1)[vec(\Sigma_2-\Sigma_1)]'\sum\nolimits_{t=k+1}^{k_0} y_t\right|$$
$$\leq \frac{2M}{T-k_0}\left\|\Sigma_2-\Sigma_1\right\|\sup_{|k_0-k|<M,k<k_0}\left\|\sum\nolimits_{t=k+1}^{k_0} y_t\right\| = O_p(\frac{1}{T})$$

Taking together, $U^M(k) \xrightarrow{d} (k_0-k)\left\|\Sigma_2-\Sigma_1\right\|^2 - 2\sum_{t=k+1}^{k_0}[vec(\Sigma_2-\Sigma_1)]'y_t$ for $|k_0-k|<M$ and $k<k_0$. Similarly, for $|k_0-k|<M$ and $k>k_0$, $U^M(k) \xrightarrow{d} (k-k_0)\left\|\Sigma_2-\Sigma_1\right\|^2 - 2\sum_{t=k_0+1}^{k}[vec(\Sigma_2-\Sigma_1)]'y_t$.

Step 3: $V^M(k) \xrightarrow{d} W^M(k-k_0)$ as $(N,T)\to\infty$ for any fixed $M<\infty$

Based on step 1 and step 2 and using Slutsky's Lemma, $V^M(k) \xrightarrow{d} W^M(k-k_0)$.

Step 4: $\arg\min V^M(k)-k_0 \xrightarrow{d} \arg\min W^M(l)$ as $(N,T)\to\infty$ for any fixed $M<\infty$.

If $W(l)$ does not have unique maximizer, then these exist $l\neq l'$ such that $W(l)=W(l')$. It's easy to see $P(W(l)=W(l'))=0$. The number of integer pairs (l,l') is countable and sum of countable zero is zero, hence the probability that $W(l)$ does not have unique maximizer is zero.

Next, for a finite dimensional vector x, $f(x)=\arg\min x$ is a continuous function, hence by continuous mapping theorem we have $\arg\min V^M(k)-k_0 \xrightarrow{d} \arg\min W^M(l)$.

By definition of convergence in distribution, for any $\epsilon>0$ and any $|j|\leq M$, there exist $N_j^*>0$ and $T_j^*>0$ such that for $N>N_j^*$ and $T>T_j^*$, $\left|P(\arg\min V^M(k)-k_0=j)-P(\arg\min W^M(l)=j)\right|<\epsilon$. Take $N^*=\max\{N_j^*,|j|\leq M\}$ and $T^*=\max\{T_j^*,|j|\leq M\}$. For $N>N^*$ and $T>T^*$, $\left|P(\arg\min V^M(k)-k_0=j)-P(\arg\min W^M(l)=j)\right|<\epsilon$ for all $|j|\leq M$.

Step 5: $\tilde{k}-k_0 \xrightarrow{d} \arg\min W(l)$ as $(N,T)\to\infty$.

Step 5.1: By Theorem 1, $\tilde{k}-k_0=O_p(1)$ as $(N,T)\to\infty$, hence for any $\epsilon>0$, there exist $M_1<\infty$, $N_1>0$ and $T_1>0$, such that for $N>N_1$ and $T>T_1$, $P(\left|\tilde{k}-k_0\right|>M_1)<\frac{\epsilon}{3}$.

Step 5.2: $\tilde{l}=\arg\min W(l)=O_p(1)$ as $(N,T)\to\infty$.

First note that $P(\min_{|l|>M} W(l) \le 0) \le P(\min_{l<-M} W_1(l) \le 0) + P(\min_{l>M} W_2(l) \le 0)$

$= P(\sup_{l<-M} \{-l \|\Sigma_2 - \Sigma_1\|^2 - 2\sum_{t=k_0+l}^{k_0} [vec(\Sigma_2 - \Sigma_1)]' y_t\} \le 0)$

$+P(\sup_{l>M} \{l \|\Sigma_2 - \Sigma_1\|^2 - 2\sum_{t=k_0+1}^{k_0+l} [vec(\Sigma_2 - \Sigma_1)]' y_t\} \le 0)$

$\le P(\sup_{l<-M} 2[vec(\Sigma_2 - \Sigma_1)]' \frac{1}{l} \sum_{t=k_0+l}^{k_0} y_t \ge \|\Sigma_2 - \Sigma_1\|^2) + P(\sup_{l>M} 2[vec(\Sigma_2 - \Sigma_1)]' \frac{1}{l} \sum_{t=k_0+1}^{k_0+l} y_t \ge \|\Sigma_2 - \Sigma_1\|^2)$

$\le P(\sup_{l<-M} \left\| \frac{1}{-l} \sum_{t=k_0+l}^{k_0} y_t \right\| \ge \frac{\|\Sigma_2 - \Sigma_1\|}{2}) + P(\sup_{l>M} \left\| \frac{1}{l} \sum_{t=k_0+1}^{k_0+l} y_t \right\| \ge \frac{\|\Sigma_2 - \Sigma_1\|}{2}) = \frac{C}{M}$ by Hajek-Renyi inequality. Hence for any $\epsilon > 0$, there exists $M_2 < \infty$ such that $P(\sup_{|l|>M_2} W(l) \le 0) < \frac{\epsilon}{3}$. Since $W(0) = 0$, $\min W(l) \le 0$, therefore $P(\left|\tilde{l}\right| > M_2) \le P(\min_{|l|>M_2} W(l) \le 0) < \frac{\epsilon}{3}$.

Step 5.3:

Take $M = \max\{M_1, M_2\}$. Based on step 4, for any $\epsilon > 0$ there exist $N_2 > 0$ and $T_2 > 0$, such that for $N > N_2$ and $T > T_2$, $\left|P(\arg\min V^M(k) - k_0 = j) - P(\arg\min W^M(l) = j)\right| < \frac{\epsilon}{3}$ for all $|j| \le M$.

Step 5.4:

Take $N^* = \max\{N_1, N_2\}$ and $T^* = \max\{T_1, T_2\}$. For any $N > N^*$ and $T > T^*$,

if $|j| > M$,

$\left|P(\tilde{k} - k_0 = j) - P(\tilde{l} = j)\right| < P(\tilde{k} - k_0 = j) + P(\tilde{l} = j) < P(\left|\tilde{k} - k_0\right| > M) + P(\left|\tilde{l}\right| > M) < \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$;

if $|j| \le M$,

$\tilde{k} - k_0 = j$ implies $\arg\min V^M(k) - k_0 = j$, hence $P(\tilde{k} - k_0 = j) \le P(\arg\min V^M(k) - k_0 = j)$,

$\arg\min V^M(k) - k_0 = j$ implies $\tilde{k} - k_0 = j$ or $\left|\tilde{k} - k_0\right| > M$,

hence $P(\arg\min V^M(k) - k_0 = j) < P(\tilde{k} - k_0 = j) + P(\left|\tilde{k} - k_0\right| > M)$,

therefore $\left|P(\tilde{k} - k_0 = j) - P(\arg\min V^M(k) - k_0 = j)\right| < P(\left|\tilde{k} - k_0\right| > M) < \frac{\epsilon}{3}$,

similarly $\left|P(\tilde{l} = j) - P(\arg\min W^M(l) = j)\right| < P(\left|\tilde{l}\right| > M) < \frac{\epsilon}{3}$,

therefore $\left|P(\tilde{k} - k_0 = j) - P(\tilde{l} = j)\right| < \left|P(\tilde{k} - k_0 = j) - P(\arg\min V^M(k) - k_0 = j)\right|$

$+ \left|P(\arg\min V^M(k) - k_0 = j) - P(\arg\min W^M(l) = j)\right| + \left|P(\hat{l} = j) - P(\arg\min W^M(l) = j)\right|$

$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon$.

Therefore, we proved that for any $\epsilon > 0$, there exist $N^* > 0$ and $T^* > 0$, such that for $N > N^*$ and $T > T^*$, $\left|P(\tilde{k} - k_0 = j) - P(\tilde{l} = j)\right| < \epsilon$ for all j. By definition, $\tilde{k} - k_0 \xrightarrow{d} \arg\min W(l)$. ■

D PROOF OF COROLLARY 1

Proof. The proof is the same as the proof of Proposition 2, Theorem 1 and Theorem 2, except for some slight modification. When $m < r+q_1$, V_{NT}, $\tilde{G}$ and J are replaced by V_{NT}^m, $\tilde{G}^m$ and J^m respectively, where V_{NT} is the diagonal matrix of the first m largest eigenvalues of $\frac{1}{NT}XX'$ in decreasing order and $\tilde{G}^m$ is $\sqrt{T}$ times the corresponding eigenvector matrix and $J^m = \frac{\Gamma'\Gamma}{N}\frac{G'\tilde{G}^m}{T}(V_{NT}^m)^{-1}$. $V_{NT}^m \xrightarrow{p} V^m$, where V^m is $m \times m$ diagonal matrix, containing the first m diagonal elements of V. $\frac{G'\tilde{G}^m}{T}$ contains the first m columns of $\frac{G'\tilde{G}}{T}$, hence $\frac{G'\tilde{G}}{T} \xrightarrow{p} \Sigma_\Gamma^{-\frac{1}{2}}\Phi V^{\frac{1}{2}}$ implies $\frac{G'\tilde{G}^m}{T} \xrightarrow{p} D$ where D contains the first m columns of $\Sigma_\Gamma^{-\frac{1}{2}}\Phi V^{\frac{1}{2}}$. Hence $D(V^m)^{-1}$ contains the first m columns of $\Sigma_\Gamma^{-\frac{1}{2}}\Phi V^{-\frac{1}{2}}$ and it follows that $J^m \xrightarrow{p} J_0^m$ where J_0^m contains the first m columns of J_0. ∎

E PROOF OF THEOREM 3

Proof. Consider the consistency of $\tilde{r}_1$. Due to symmetry, the consistency of $\tilde{r}_2$ can be established similarly. What we need to show is: for any $\epsilon > 0$, $P(\tilde{r}_1 \neq r_1) < \epsilon$ for large (N,T). Based on $\left|\tilde{k} - k_0\right| = O_p(1)$, we have for any $\epsilon > 0$, there exist $M > 0$ such that $P(\left|\tilde{k} - k_0\right| > M) < \epsilon$ for all (N,T). Based on this M, $P(\tilde{r}_1 \neq r_1)$ can be decomposed as

$$P(\tilde{r}_1 \neq r_1) = P(\tilde{r}_1 \neq r_1, \left|\tilde{k} - k_0\right| > M) + P(\tilde{r}_1 \neq r_1, k_0 - M \leq \tilde{k} \leq k_0) + P(\tilde{r}_1 \neq r_1, k_0+1 \leq \tilde{k} \leq k_0+M).$$

The first term is less than $P(\left|\tilde{k} - k_0\right| > M)$, hence less than ϵ for all (N,T). The second term can be further decomposed as

$$P(\tilde{r}_1 \neq r_1, k_0 - M \leq \tilde{k} \leq k_0) = \sum_{k=k_0-M}^{k_0} P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k),$$

where $P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k)$ denotes the joint probability of $\tilde{k} = k$ and $\tilde{r}_1(k) \neq r_1$ and $\tilde{r}_1(k)$ denotes the estimated number of pre-break factors using subsample $t = 1, ..., k$. Obviously, $P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k) \leq P(\tilde{r}_1(k) \neq r_1)$, hence the second term is less than $\sum_{k=k_0-M}^{k_0} P(\tilde{r}_1(k) \neq r_1)$. Furthermore, since for each $k \in [k_0 - M, k_0]$, the factor loadings in the pre-break subsample are stable, $P(\tilde{r}_1(k) \neq r_1) \leq \frac{\epsilon}{M+1}$ for large (N,T). Therefore, the second term is less than $\sum_{k=k_0-M}^{k_0} \frac{\epsilon}{M+1} = \epsilon$ for large (N,T).

The argument for the second term also applies to the third term, except for some modifications. First, the third can be decomposed similarly as

$$P(\tilde{r}_1 \neq r_1, k_0 + 1 \leq \tilde{k} \leq k_0 + M) = \sum_{k=k_0+1}^{k_0+M} P(\tilde{r}_1(k) \neq r_1, \tilde{k} = k) \leq \sum_{k=k_0+1}^{k_0+M} P(\tilde{r}_1(k) \neq r_1),$$

hence it remains to show for each $k \in [k_0+1, k_0+M]$, $P(\tilde{r}_1(k) \neq r_1) \leq \frac{\epsilon}{M}$ for large (N,T). Unlike the second term, when $k \in [k_0+1, k_0+M]$ the factor loadings of the pre-break subsample $t = 1, ..., k$ has a break at $t = k_0$, hence results already established in previous literature for stable model is not directly applicable. To overcome this difficulty, we treat change in factor loadings as an extra error term such that $x_{it} = f_t'\lambda_{02,i} + e_{it} = f_t'\lambda_{01,i} + e_{it} + w_{it} = a_{it} + w_{it}$, where $a_{it} = f_t'\lambda_{01,i} + e_{it}$, $w_{it} = 0$ for $1 \leq t \leq k_0$ and $w_{it} = f_t'\lambda_{02,i} - f_t'\lambda_{01,i}$ for $t \geq k_0+1$. In other words, when $k \geq k_0+1$ the pre-break subsample $t = 1, ..., k$ can be regarded as having stable factor loadings and an extra error term in observations $t = k_0+1, ...k$. In matrix form, we have $X(k) = A(k) + W(k)$, where $X(k)$, $A(k)$ and $W(k)$ are all $k \times N$ matrix. Define ω_j^k, α_j^k and β_j^k as the j-th largest eigenvalue of $\frac{1}{Nk}X(k)X'(k)$, $\frac{1}{Nk}A(k)A'(k)$ and $\frac{1}{Nk}W(k)W'(k)$ respectively. By Weyl's inequality for singular values, the perturbation effect of the extra error matrix $W(k)$ on the eigenvalues of $A(k)$ is

$$\sqrt{\alpha_j^k} - \sqrt{\beta_1^k} \leq \sqrt{\omega_j^k} \leq \sqrt{\alpha_j^k} + \sqrt{\beta_1^k}, \tag{31}$$

hence $(\sqrt{\omega_j^k} - \sqrt{\alpha_j^k})^2 \leq \beta_1^k$. Since the number of nonzero elements in the $k \times N$ matrix $W(k)$ is only $(k-k_0) \times N$ and $k - k_0 \leq M$, simple calculation shows that

$$\begin{aligned} \beta_1^k &\leq tr(\frac{1}{Nk}W(k)W'(k)) = \frac{1}{Nk}\sum_{i=1}^N \sum_{t=k_0+1}^k w_{it}^2 \\ &\leq 2\frac{1}{Nk_0}\sum_{i=1}^N \sum_{t=k_0+1}^k \|f_t\|^2 (\|\lambda_{01,i}\|^2 + \|\lambda_{02,i}\|^2) \\ &\leq 8\frac{1}{k_0}\sum_{t=k_0+1}^{k_0+M} \|f_t\|^2 \|\bar{\lambda}\|^2 = O_p(\frac{1}{T}). \end{aligned} \tag{32}$$

In addition, according to Bai and Ng (2002), $\alpha_j^k = \nu_j + o_p(1)$ for $j \leq r_1$, where ν_j is the j-th largest eigenvalue of $\Sigma_F \Sigma_{\Lambda_{01}}$, and $\alpha_j^k = O_p(\frac{1}{\delta_{NT}^2})$ for $j > r_1$. It follows that $\omega_j^k = \alpha_j^k + 2\sqrt{\alpha_j^k} O_p(\frac{1}{\sqrt{T}}) + O_p(\frac{1}{T}) = \nu_j + o_p(1)$ for $j \leq r_1$, and $\omega_j^k = O_p(\frac{1}{\delta_{NT}^2}) + O_p(\frac{1}{\delta_{NT}})O_p(\frac{1}{\sqrt{T}}) + O_p(\frac{1}{T}) = O_p(\frac{1}{\delta_{NT}^2})$ for $j > r_1$. This implies that the estimator of number of factors using Bai and Ng (2002) or other methods based on the sample $X(k)$ is still consistent for $k \in [k_0+1, k_0+M]$, hence $P(\tilde{r}_1(k) \neq r_1) \leq \frac{\epsilon}{M}$ for large (N,T). ■

F PROOF OF PROPOSITION 3

Proof. The proof is similar to Theorem 3.

$$\begin{aligned} \beta_1^T &\leq tr(\frac{1}{NT}W(T)W'(T)) = \frac{1}{NT}\sum_{i=1}^N \sum_{t=k_0+1}^T w_{it}^2 \\ &\leq \frac{1}{NT}\sum_{i=1}^N \sum_{t=k_0+1}^T \|f_t\|^2 \|\lambda_{02,i} - \lambda_{01,i}\|^2 \\ &= (\frac{1}{T}\sum_{t=k_0+1}^T \|f_t\|^2)(\frac{1}{N}\sum_{i=1}^N \|\lambda_{02,i} - \lambda_{01,i}\|^2) = O_p(\frac{1}{\delta_{NT}^c}). \end{aligned} \tag{33}$$

By Weyl's inequality for singular values, $\sqrt{\alpha_j^T} - \sqrt{\beta_1^T} \le \sqrt{\omega_j^T} \le \sqrt{\alpha_j^T} + \sqrt{\beta_1^T}$, hence $(\sqrt{\omega_j^T} - \sqrt{\alpha_j^T})^2 \le \beta_1^T = O_p(\frac{1}{\delta_{NT}^{\frac{c}{2}}})$. It follows that $\omega_j^T = \alpha_j^T + 2\sqrt{\alpha_j^T} O_p(\frac{1}{\delta_{NT}^{\frac{c}{2}}}) + O_p(\frac{1}{\delta_{NT}^c}) = \nu_j + o_p(1)$ for $j \le r_1$, and $\omega_j^T = O_p(\frac{1}{\delta_{NT}^2}) + O_p(\frac{1}{\delta_{NT}}) O_p(\frac{1}{\delta_{NT}^{\frac{c}{2}}}) + O_p(\frac{1}{\delta_{NT}^c}) = O_p(\frac{1}{\delta_{NT}^c})$ for $j > r_1$ when $c < 2$. ■

G PROOF OF THEOREM 4

Proof. Again by symmetry, we only need to show the first half.

To show $\frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$, we need to show for any $\epsilon > 0$, there exist $C > 0$ such that $P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C) < \epsilon$ for all (N,T). First, based on $\left|\tilde{k} - k_0\right| = O_p(1)$ we can choose $M > 0$ such that $P(\left|\tilde{k} - k_0\right| > M) < \frac{\epsilon}{2}$ for the given ϵ. Next, note that $P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C) = P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C, \left|\tilde{k} - k_0\right| > M) + \sum_{k=k_0-M}^{k_0+M} P(\delta_{NT}^2 \frac{1}{\tilde{k}} \sum_{t=1}^{\tilde{k}} \left\| \hat{f}_t^u(\tilde{k}) - H_1^{u\prime}(\tilde{k}) f_t \right\|^2 > C, \tilde{k} = k) \le P(\left|\tilde{k} - k_0\right| > M) + \sum_{k=k_0-M}^{k_0+M} P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C) \le \frac{\epsilon}{2} + \sum_{k=k_0-M}^{k_0+M} P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C)$. If we can show $\frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ for each $k \in [k_0 - M, k_0 + M]$, then for the given ϵ and for each $k \in [k_0 - M, k_0 + M]$, we can take $C(k) > 0$ such that $P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C(k)) < \frac{\epsilon}{2(2M+1)}$ for all (N,T). Take $C = \max_{k \in [k_0 - M, k_0 + M]} C(k)$, then $P(\delta_{NT}^2 \frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 > C) \le \frac{\epsilon}{2} + \sum_{k=k_0-M}^{k_0+M} \frac{\epsilon}{2(2M+1)} = \epsilon$ for all (N,T), hence it remains to show $\frac{1}{k} \sum_{t=1}^{k} \left\| \hat{f}_t^u(k) - H_1^{u\prime}(k) f_t \right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ for each $k \in [k_0 - M, k_0 + M]$.

First consider the case $k_0 - M \le k \le k_0$. In this case, factor loadings are stable and $k_0 - M \le k$ guarantees $k \to \infty$ as $k_0 \to \infty$, hence Theorem 1 of Bai and Ng (2002) is applicable.

Next consider the case $k_0 + 1 \le k \le k_0 + M$. Following the same notation as proof of Theorem 3 and define $E(k) = (e_1, ..., e_k)'$, we have $X(k) = A(k) + W(k) = F_1(k)\Lambda_{01}' + E(k) + W(k)$, thus

$$\begin{aligned} X(k)X'(k) &= F_1(k)\Lambda_{01}'\Lambda_{01}F_1'(k) + F_1(k)\Lambda_{01}'[E(k) + W(k)]' + [E(k) + W(k)]\Lambda_{01}F_1'(k) \\ &\quad + [E(k) + W(k)][E(k) + W(k)]'. \end{aligned} \tag{34}$$

It follows that

$$\begin{aligned}
\hat{f}_t^u(k) - H_1^{w\prime}(k) f_t &= \frac{1}{Nk}[\tilde{F}_1^{w\prime}(k)F_1(k)\Lambda_{01}' e_t + \tilde{F}_1^{w\prime}(k)E(k)\Lambda_{01} f_t + \tilde{F}_1^{w\prime}(k)E(k)e_t \\
&\quad + \tilde{F}_1^{w\prime}(k)F_1(k)\Lambda_{01}' w_t + \tilde{F}_1^{w\prime}(k)W(k)\Lambda_{01} f_t + \tilde{F}_1^{w\prime}(k)W(k)w_t \\
&\quad + \tilde{F}_1^{w\prime}(k)E(k)w_t + \tilde{F}_1^{w\prime}(k)W(k)e_t] \\
&= Q_{1,t}(k) + Q_{2,t}(k) + Q_{3,t}(k) + Q_{4,t}(k) + Q_{5,t}(k) + Q_{6,t}(k) \\
&\quad + Q_{7,t}(k) + Q_{8,t}(k), \qquad (35)
\end{aligned}$$

and $\frac{1}{k}\sum_{t=1}^{k}\left\|\hat{f}_t^u(k) - H_1^{w\prime}(k) f_t\right\|^2 \leq 8\sum_{m=1}^{8}\frac{1}{k}\sum_{t=1}^{k}\|Q_{m,t}(k)\|^2$. Following the same procedure as proof of Theorem 1 in Bai and Ng (2002), it can be shown $\frac{1}{k}\sum_{t=1}^{k}\|Q_{m,t}(k)\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ for $m = 1, 2, 3$. Next, noting that $w_{it} = 0$ for $1 \leq t \leq k_0$,

$$\begin{aligned}
\frac{1}{k}\sum_{t=1}^{k}\|Q_{4,t}(k)\|^2 &= \frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{Nk}\tilde{F}_1^{w\prime}(k)F_1(k)\Lambda_{01}' w_t\right\|^2 \\
&\leq \frac{1}{k}\sum_{t=1}^{k}(\frac{1}{k}\sum_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{k}\sum_{s=1}^{k}\|f_s\|^2)\left\|\frac{1}{N}\Lambda_{01}' w_t\right\|^2 \\
&\leq (\frac{1}{k}\sum_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{k}\sum_{s=1}^{k}\|f_s\|^2)(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{01,i}\|^2) \\
&\quad (\frac{1}{k}\sum_{t=1}^{k}\frac{1}{N}\sum_{i=1}^{N}\|w_{it}\|^2) \\
&\leq (\frac{1}{k}\sum_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{k}\sum_{s=1}^{k}\|f_s\|^2)(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{01,i}\|^2) \\
&\quad (\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{01,i} - \lambda_{02,i}\|^2)(\frac{1}{k}\sum_{t=k_0+1}^{k}\|f_t\|^2) \\
&= O_p(1)O_p(1)O(1)O(1)O_p(\frac{1}{T}) = O_p(\frac{1}{T}),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{k}\sum_{t=1}^{k}\|Q_{5,t}(k)\|^2 &= \frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{Nk}\tilde{F}_1^{w\prime}(k)W(k)\Lambda_{01} f_t\right\|^2 \\
&\leq \frac{1}{k}\sum_{t=1}^{k}(\frac{1}{k}\sum_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{N^2}\frac{1}{k}\sum_{s=1}^{k}\left\|w_s'\Lambda_{01} f_t\right\|^2) \\
&\leq (\frac{1}{k}\sum_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{01,i}\|^2)(\frac{1}{k}\sum_{t=1}^{k}\|f_t\|^2) \\
&\quad (\frac{1}{N}\sum_{i=1}^{N}\|\lambda_{01,i} - \lambda_{02,i}\|^2)(\frac{1}{k}\sum_{s=k_0+1}^{k}\|f_s\|^2) \\
&= O_p(1)O(1)O_p(1)O(1)O_p(\frac{1}{T}) = O_p(\frac{1}{T}),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{k}\sum\nolimits_{t=1}^{k}\|Q_{6,t}(k)\|^2 &= \frac{1}{k}\sum\nolimits_{t=1}^{k}\left\|\frac{1}{Nk}\tilde{F}_1^{u\prime}(k)W(k)w_t\right\|^2 \\
&\leq (\frac{1}{k}\sum\nolimits_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)\frac{1}{N^2}(\frac{1}{k}\sum\nolimits_{s=1}^{k}\|w_s\|^2)(\frac{1}{k}\sum\nolimits_{t=1}^{k}\|w_t\|^2) \\
&\leq (\frac{1}{k}\sum\nolimits_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{N}\sum\nolimits_{i=1}^{N}\|\lambda_{01,i}-\lambda_{02,i}\|^2)^2 \\
&\quad (\frac{1}{k}\sum\nolimits_{s=k_0+1}^{k}\|f_s\|^2)(\frac{1}{k}\sum\nolimits_{t=k_0+1}^{k}\|f_t\|^2) \\
&= O_p(1)O(1)O_p(\frac{1}{T})O_p(\frac{1}{T}) = O_p(\frac{1}{T^2}),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{k}\sum\nolimits_{t=1}^{k}\|Q_{7,t}(k)\|^2 &= \frac{1}{k}\sum\nolimits_{t=1}^{k}\left\|\frac{1}{Nk}\tilde{F}_1^{u\prime}(k)E(k)w_t\right\|^2 \\
&\leq (\frac{1}{k}\sum\nolimits_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{k}\frac{1}{N}\sum\nolimits_{s=1}^{k}\sum\nolimits_{i=1}^{N}e_{is}^2) \\
&\quad (\frac{1}{k}\sum\nolimits_{t=k_0+1}^{k}\|f_t\|^2)(\frac{1}{N}\sum\nolimits_{i=1}^{N}\|\lambda_{01,i}-\lambda_{02,i}\|^2) \\
&= O_p(1)O_p(1)O_p(\frac{1}{T})O(1) = O_p(\frac{1}{T}),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{k}\sum\nolimits_{t=1}^{k}\|Q_{8,t}(k)\|^2 &= \frac{1}{k}\sum\nolimits_{t=1}^{k}\left\|\frac{1}{Nk}\tilde{F}_1^{u\prime}(k)W(k)e_t\right\|^2 \\
&\leq \frac{1}{k}\sum\nolimits_{t=1}^{k}(\frac{1}{k}\sum\nolimits_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)\frac{1}{N^2}(\frac{1}{k}\sum\nolimits_{s=1}^{k}\|w_s'e_t\|^2) \\
&\leq (\frac{1}{k}\sum\nolimits_{s=1}^{k}\left\|\tilde{f}_s^u(k)\right\|^2)(\frac{1}{k}\sum\nolimits_{t=k_0+1}^{k}\|f_s\|^2) \\
&\quad (\frac{1}{N}\sum\nolimits_{i=1}^{N}\|\lambda_{01,i}-\lambda_{02,i}\|^2)(\frac{1}{k}\frac{1}{N}\sum\nolimits_{t=1}^{k}\sum\nolimits_{i=1}^{N}e_{it}^2) \\
&= O_p(1)O_p(\frac{1}{T})O(1)O_p(1) = O_p(\frac{1}{T}),
\end{aligned}$$

hence $\frac{1}{k}\sum_{t=1}^{k}\|Q_{m,t}(k)\|^2 = O_p(\frac{1}{\delta_{NT}^2})$ for $m = 4,5,6,7,8$, and thus $\frac{1}{k}\sum_{t=1}^{k}\left\|\hat{f}_t^u(k) - H_1^{u\prime}(k)f_t\right\|^2 = O_p(\frac{1}{\delta_{NT}^2})$. ■

H PROOF OF LEMMAS

Lemma 1 *Under Assumptions 1-4,* $\frac{1}{T}\sum_{t=1}^{T}\|\tilde{g}_t - J'g_t\|^2 = O_p(\frac{1}{\delta_{NT}^2})$.

Proof. This Lemma is Theorem 1 of Bai and Ng (2002) for the equivalent model, therefore it suffices to verify Assumptions A-D of Bai and Ng (2002).

Assumption A:

By Assumption 1, $E\|g_t\|^4 \leq \max\{\|A\|^4, \|B\|^4\}E\|f_t\|^4 < M < \infty$, $\frac{1}{T}\sum_{t=1}^{T}g_tg_t' = \tau_0\frac{1}{k_0}\sum_{t=1}^{k_0}Af_tf_t'A' + (1-\tau_0)\frac{1}{T-k_0}\sum_{t=k_0+1}^{T}Bf_tf_t'B' \xrightarrow{p} \tau_0 A\Sigma_F A' + (1-\tau_0)B\Sigma_F B' = \Sigma_G$ and Σ_G is positive definite.

Assumption B:

By Assumption 2, $\|\gamma_i\| \leq \left\|(\lambda'_{0,i}, \lambda'_{1,i}, \lambda'_{2,i})'\right\| = (\|\lambda_{0,i}\|^2 + \|\lambda_{1,i}\|^2 + \|\lambda_{2,i}\|^2)^{\frac{1}{2}} \leq \sqrt{3}\bar{\lambda} < \infty$ and $\left\|\frac{1}{N}\Gamma'\Gamma - \Sigma_\Gamma\right\| \to 0$ for some positive definite matrix Σ_Γ.

Assumption C:

Assumption 3 is identical to Assumption C.

Assumption D:

$E(\frac{1}{N}\sum_{i=1}^N \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^T g_t e_{it}\right\|^2) \leq 2\|A\|^2 E(\frac{1}{N}\sum_{i=1}^N \left\|\frac{1}{\sqrt{T}}\sum_{t=1}^{k_0} Af_t e_{it}\right\|^2)$
$+2\|B\|^2 E(\frac{1}{N}\sum_{i=1}^N \left\|\frac{1}{\sqrt{T}}\sum_{t=k_0+1}^T f_t e_{it}\right\|^2) \leq 2\tau_0 M + 2(1-\tau_0)M = 2M.$ ■

Lemma 2 *Under Assumptions 1-4 and 7,* $\|J - J_0\| = o_p(1)$.

Proof. This Lemma follows from Proposition 1 of Bai (2003). Assumptions A-D is verified in Lemma 1, Assumption G is identical to Assumption 7. ■

Lemma 3 *Under Assumptions 1-8,*

(1) Hajek-Renyi inequality applies to the process $\{y_t, t = 1, ..., k_0\}$, $\{y_t, t = k_0, ..., 1\}$, $\{y_t, t = k_0 + 1, ..., T\}$ *and* $\{y_t, t = T, ..., k_0 + 1\}$,

(2) $\sup_{k \leq k_0} \frac{1}{k}\sum_{t=1}^k \|g_t\|^2 = O_p(1)$, $\sup_{k \geq k_0} \frac{1}{T-k}\sum_{t=k+1}^T \|g_t\|^2 = O_p(1)$, $\sup_{k < k_0} \frac{1}{k_0-k}\sum_{t=k+1}^{k_0} \|g_t\|^2 = O_p(1)$ *and* $\sup_{k > k_0} \frac{1}{k-k_0}\sum_{t=k_0+1}^k \|g_t\|^2 = O_p(1)$.

Proof. (1) $P(\sup_{m \leq k \leq k_0} c_k \left\|\sum_{t=1}^k y_t\right\| > M) = P(\sup_{m \leq k \leq k_0} c_k \left\|J'_0 A[\sum_{t=1}^k (f_t f'_t - \Sigma_F)]A' J_0\right\| > M)$

$\leq P(\|J'_0 A\|^2 \sup_{m \leq k \leq k_0} c_k \left\|\sum_{t=1}^k \epsilon_t\right\| > M) \leq \frac{C}{M^2}(mc_m^2 + \sum_{k=m+1}^{k_0} c_k^2)$, where the last inequality follows from Hajek-Renyi inequality for process $\{\epsilon_t, t = 1, ..., k_0\}$. Other processes can be proved similarly.

(2) $\sup_{k \leq k_0} \frac{1}{k}\sum_{t=1}^k \|g_t\|^2 \leq \|A\|^2 E\|f_t\|^2 + \|A\|^2 \sup_{k \leq k_0} \frac{1}{k}\sum_{t=1}^k (\|f_t\|^2 - E\|f_t\|^2)$, where $E\|f_t\|^2 = tr\Sigma_F$. Define $D_k = \frac{1}{k}\sum_{t=1}^k (f_t f'_t - \Sigma_F)$, then $\left|\frac{1}{k}\sum_{t=1}^k (\|f_t\|^2 - E\|f_t\|^2)\right| = |trD_k| \leq \sqrt{r+q_1}(trD_k^2)^{\frac{1}{2}} = \sqrt{r+q_1}\|D_k\|$, it follows $\left|\sup_{k \leq k_0} \frac{1}{k}\sum_{t=1}^k (\|f_t\|^2 - E\|f_t\|^2)\right| \leq \sup_{k \leq k_0} \left|\frac{1}{k}\sum_{t=1}^k (\|f_t\|^2 - E\|f_t\|^2)\right| \leq \sqrt{r+q_1}$ $\sup_{k \leq k_0} \left\|\frac{1}{k}\sum_{t=1}^k \epsilon_t\right\|$, which is $O_p(1)$ by Hajek-Renyi inequality. Thus $\sup_{k \leq k_0} \frac{1}{k}\sum_{t=1}^k \|g_t\|^2 \leq \|A\|^2 E\|f_t\|^2 + \|A\|^2 O_p(1) = O_p(1)$. Other terms can be proved similarly. ■

Lemma 4 *General Hajek-Renyi inequality (Theorem 1.1 of Fazekas and Klesov (2001)):*

Let $\beta_1, \beta_2, ..., \beta_n$ be a sequence of nondecreasing positive numbers. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be a sequence of nonnegative numbers. Let r be a fixed positive number. For the partial sum process $S_l = \sum_{k=1}^{l} X_k$, assume for each m with $1 \leq m \leq n$, $E(\sup_{1\leq l\leq m} |S_l|^r) \leq \sum_{l=1}^{m} \alpha_l$, then $E(\sup_{1\leq l\leq n} \left|\frac{S_l}{\beta_l}\right|^r) \leq 4\sum_{l=1}^{n} \frac{\alpha_l}{\beta_l^r}$.

Note that no dependence structure on $\{X_k, k = 1, ...\}$ is assumed.

Lemma 5 *Under Assumptions 1-8 and 10,*

(1) $\sup_{k\in D, k\leq k_0} \left\| \frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t - J'g_t)g_t'J \right\| = O_p(\frac{1}{\delta_{NT}})$,

(2) $\sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t - J'g_t)g_t'J \right\| = O_p(\frac{1}{\delta_{NT}})$,

(3) $\sup_{k\in D^c, k<k_0} \left\| \frac{1}{k_0-k}\sum_{t=k+1}^{k_0}(\tilde{g}_t - J'g_t)g_t'J \right\| = O_p(\frac{1}{\delta_{NT}})$,

(4) $\sup_{k\in D, k<k_0} \left\| \frac{1}{k_0-k}\sum_{t=k+1}^{k_0}(\tilde{g}_t - J'g_t)g_t'J \right\| = O_p(\frac{1}{\delta_{NT}})$,

(5) $\sup_{k\in D, k<k_0} \left\| \frac{1}{k_0-k}\sum_{t=k+1}^{k_0}(\tilde{g}_t - J'g_t)(\tilde{g}_t - J'g_t)' \right\| = O_p(\frac{1}{\delta_{NT}^2})$,

(6) $\sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t - J'g_t)(\tilde{g}_t - J'g_t)' \right\| = O_p(\frac{1}{\delta_{NT}^2})$,

(7) $\sup_{k\leq k_0} \left\| \frac{1}{T-k}\sum_{t=k+1}^{T}(\tilde{g}_t - J'g_t)g_t'J \right\| = O_p(\frac{1}{\delta_{NT}})$.

Proof. We will prove parts (2), (5) and (7). Proof of parts (1), (3) and (4) is similar to part (2), proof of part (6) is similar to part (5). First consider part (2).

$$\begin{aligned}
& \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t - J'g_t)g_t'J \right\| \\
= \; & \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{k}\sum_{t=1}^{k} V_{NT}^{-1}\frac{1}{T}(\sum_{s=1}^{T}\tilde{g}_s\gamma_N(s,t) + \sum_{s=1}^{T}\tilde{g}_s\zeta_{st} + \sum_{s=1}^{T}\tilde{g}_s\eta_{st} + \sum_{s=1}^{T}\tilde{g}_s\xi_{st})g_t'J \right\| \\
\leq \; & (\sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T}(\tilde{g}_s - J'g_s)g_t'\gamma_N(s,t) \right\| \\
& + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} J'g_sg_t'\gamma_N(s,t) \right\| \\
& + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T}(\tilde{g}_s - J'g_s)g_t'\zeta_{st} \right\| + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} J'g_sg_t'\zeta_{st} \right\| \\
& + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T}(\tilde{g}_s - J'g_s)g_t'\eta_{st} \right\| + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} J'g_sg_t'\eta_{st} \right\| \\
& + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T}(\tilde{g}_s - J'g_s)g_t'\xi_{st} \right\| \\
& + \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} J'g_sg_t'\xi_{st} \right\|) \left\| V_{NT}^{-1} \right\| \left\| J \right\| \\
= \; & (I + II + III + IV + V + VI + VII + VIII) \left\| V_{NT}^{-1} \right\| \left\| J \right\|. \qquad (36)
\end{aligned}$$

Consider the eight terms one by one.

$$\begin{aligned}
I &\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}(\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k} g_t'\gamma_N(s,t)\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}[(\frac{1}{k}\sum\nolimits_{t=1}^{k}\|g_t\|^2)(\frac{1}{T}\frac{1}{k}\sum\nolimits_{s=1}^{T}\sum\nolimits_{t=1}^{k}|\gamma_N(s,t)|^2)]^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum\nolimits_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{T}\frac{1}{k}\sum\nolimits_{s=1}^{T}\sum\nolimits_{t=1}^{k}|\gamma_N(s,t)|^2)^{\frac{1}{2}} \\
&= O_p(\frac{1}{\delta_{NT}})O_p(1)O_p(\frac{1}{\sqrt{T}}),
\end{aligned}$$

where last equality follows from Lemma 1, Lemma 3 and $\sup_{k\in D^c, k\leq k_0}\frac{1}{T}\frac{1}{k}\sum_{s=1}^{T}\sum_{t=1}^{k}|\gamma_N(s,t)|^2 \leq \frac{1}{T}\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}(\sum_{s=1}^{T} M\,|\gamma_N(s,t)|) \leq \frac{1}{T}M^2$ by part (2) of Assumption 3 and part (1) of Assumption 5.

$$\begin{aligned}
II &\leq \|J\| \sup_{k\in D^c, k\leq k_0}\frac{1}{T}\sum\nolimits_{s=1}^{T}\|g_s\|\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k} g_t'\gamma_N(s,t)\right\| \\
&\leq \|J\|(\frac{1}{T}\sum\nolimits_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}(\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k} g_t'\gamma_N(s,t)\right\|^2)^{\frac{1}{2}} \\
&\leq \|J\|(\frac{1}{T}\sum\nolimits_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum\nolimits_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{T}\frac{1}{k}\sum\nolimits_{s=1}^{T}\sum\nolimits_{t=1}^{k}|\gamma_N(s,t)|^2)^{\frac{1}{2}} \\
&= O_p(1)O_p(1)O_p(1)O_p(\frac{1}{\sqrt{T}}),
\end{aligned}$$

where the last equality follows from Lemma 2, Assumption 1, Lemma 3 and $\sup_{k\in D^c, k\leq k_0}\frac{1}{T}\frac{1}{k}\sum_{s=1}^{T}\sum_{t=1}^{k}$ $|\gamma_N(s,t)|^2 = O_p(\frac{1}{T})$ as explained above.

$$\begin{aligned}
III &\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}(\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k} g_t'\frac{1}{N}\sum\nolimits_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \frac{1}{\sqrt{N}}(\frac{1}{T}\sum\nolimits_{s=1}^{T}\sup_{k\in D^c, k\leq k_0}\left\|\frac{1}{k}\frac{1}{\sqrt{N}}\sum\nolimits_{t=1}^{k}\sum\nolimits_{i=1}^{N} g_t'[e_{is}e_{it} - E(e_{is}e_{it})]\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum\nolimits_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} \frac{1}{\sqrt{N}} \\
&\quad (\frac{1}{T}\sum\nolimits_{s=1}^{T}\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum\nolimits_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum\nolimits_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right\|^2)^{\frac{1}{2}} \\
&= O_p(\frac{1}{\delta_{NT}})O_p(1)\frac{1}{\sqrt{N}}O_p(1).
\end{aligned}$$

$$
\begin{aligned}
IV &\leq \|J\| \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} g_s g_t' \frac{1}{N}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right\| \\
&\leq \|J\| (\sup_{k\in D^c, k\leq k_0} \frac{1}{k}\sum_{t=1}^{k} \left\| \frac{1}{NT}\sum_{s=1}^{T}\sum_{i=1}^{N} g_s[e_{is}e_{it}-E(e_{is}e_{it})]\right\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0} \frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} \\
&\leq \|J\| \frac{1}{\sqrt{N}} (\frac{1}{T}\sum_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0} \frac{1}{k}\sum_{t=1}^{k}\frac{1}{T}\sum_{s=1}^{T}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it}-E(e_{is}e_{it})]\right|^2)^{\frac{1}{2}} \\
&\quad (\sup_{k\in D^c, k\leq k_0} \frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} \\
&= O_p(1)\frac{1}{\sqrt{N}}O_p(1)O_p(1)O_p(1) = O_p(\frac{1}{\sqrt{N}}),
\end{aligned}
$$

where the last equalities follow from part (1) of Assumption 10.

$$
\begin{aligned}
V &\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0} (\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{k}\sum_{t=1}^{k}(\frac{1}{N}\sum_{i=1}^{N} g_s'\gamma_i e_{it})g_t'\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} (\frac{1}{T}\sum_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} \frac{1}{\sqrt{N}} (\sup_{k\in D^c, k\leq k_0}\left\|\frac{1}{k}\frac{1}{\sqrt{N}}\sum_{t=1}^{k}\sum_{i=1}^{N}\gamma_i e_{it} g_t'\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} (\frac{1}{T}\sum_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} \frac{1}{\sqrt{N}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} \\
&\quad (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{it}\right\|^2)^{\frac{1}{2}} \\
&= O_p(\frac{1}{\delta_{NT}})O_p(1)\frac{1}{\sqrt{N}}O_p(1)O_p(1)
\end{aligned}
$$

$$
\begin{aligned}
VI &\leq \|J\| \sup_{k\in D^c, k\leq k_0}\left\|\frac{1}{T}\frac{1}{k}\sum_{t=1}^{k}\sum_{s=1}^{T} g_s(\frac{1}{N}\sum_{i=1}^{N} g_s'\gamma_i e_{it})g_t'\right\| \\
&\leq \|J\| \left\|\frac{1}{T}\sum_{s=1}^{T} g_s g_s'\right\| \frac{1}{\sqrt{N}} \sup_{k\in D^c, k\leq k_0}\left\|\frac{1}{k}\frac{1}{\sqrt{N}}\sum_{t=1}^{k}\sum_{i=1}^{N}\gamma_i e_{it}g_t'\right\| \\
&\leq \|J\| \left\|\frac{1}{T}\sum_{s=1}^{T} g_s g_s'\right\| \frac{1}{\sqrt{N}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{it}\right\|^2)^{\frac{1}{2}} \\
&= O_p(1)O_p(1)\frac{1}{\sqrt{N}}O_p(1)O_p(1),
\end{aligned}
$$

where the last equalities follow from part (2) of Assumption 10.

$$
\begin{aligned}
VII &\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}(\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{k}\sum_{t=1}^{k} g_t'(\frac{1}{N}\sum_{i=1}^{N} g_t'\gamma_i e_{is})\right\|^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} \sup_{k\in D^c, k\leq k_0}(\frac{1}{T}\sum_{s=1}^{T}(\frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2\left\|\frac{1}{N}\sum_{i=1}^{N}\gamma_i e_{is}\right\|)^2)^{\frac{1}{2}} \\
&\leq (\frac{1}{T}\sum_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2)^{\frac{1}{2}} (\sup_{k\in D^c, k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\|g_t\|^2)\frac{1}{\sqrt{N}}(\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{is}\right\|^2)^{\frac{1}{2}} \\
&= O_p(\frac{1}{\delta_{NT}})O_p(1)\frac{1}{\sqrt{N}}O_p(1).
\end{aligned}
$$

$$
\begin{aligned}
VIII &\leq \|J\| \sup_{k\in D^c, k\leq k_0} \left\| \frac{1}{T}\frac{1}{k}\sum\nolimits_{t=1}^{k}\sum\nolimits_{s=1}^{T} g_s g_t' (\frac{1}{N}\sum\nolimits_{i=1}^{N} g_t' \gamma_i e_{is}) \right\| \\
&\leq \|J\| \sup_{k\in D^c, k\leq k_0} \frac{1}{T}\frac{1}{k}\sum\nolimits_{t=1}^{k}\sum\nolimits_{s=1}^{T} \|g_s\| \|g_t\|^2 \left\| \frac{1}{N}\sum\nolimits_{i=1}^{N} \gamma_i e_{is} \right\| \\
&\leq \|J\| (\sup_{k\in D^c, k\leq k_0} \frac{1}{k}\sum\nolimits_{t=1}^{k} \|g_t\|^2)(\frac{1}{T}\sum\nolimits_{s=1}^{T}\|g_s\|^2)^{\frac{1}{2}} \frac{1}{\sqrt{N}} (\frac{1}{T}\sum\nolimits_{s=1}^{T} \left\| \frac{1}{\sqrt{N}}\sum\nolimits_{i=1}^{N} \gamma_i e_{is} \right\|^2)^{\frac{1}{2}} \\
&= O_p(1)O_p(1)O_p(1)\frac{1}{\sqrt{N}}O_p(1),
\end{aligned}
$$

where the equalities follow from $E(\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{is}\right\|^2) \leq M$, which follows from part (ii) of Lemma 1 in Bai and Ng (2002).

Next consider part (5).

$$
\begin{aligned}
&\sup_{k\in D, k<k_0} \left\| \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} (\tilde{g}_t - J'g_t)(\tilde{g}_t - J'g_t)' \right\| \\
\leq\; & \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}(\sum\nolimits_{s=1}^{T}\tilde{g}_s\gamma_N(s,t) + \sum\nolimits_{s=1}^{T}\tilde{g}_s\zeta_{st} + \sum\nolimits_{s=1}^{T}\tilde{g}_s\eta_{st} + \sum\nolimits_{s=1}^{T}\tilde{g}_s\xi_{st}) \right\|^2 \left\|V_{NT}^{-1}\right\|^2 \\
\leq\; & 4\sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} (\left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\gamma_N(s,t) \right\|^2 + \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\zeta_{st} \right\|^2 \\
& + \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\eta_{st} \right\|^2 + \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}\tilde{g}_s\xi_{st} \right\|^2) \left\|V_{NT}^{-1}\right\|^2 \\
\leq\; & 8(\sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}(\tilde{g}_s - J'g_s)\gamma_N(s,t) \right\|^2 \\
& + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T} J'g_s\gamma_N(s,t) \right\|^2 \\
& + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}(\tilde{g}_s - J'g_s)\zeta_{st} \right\|^2 + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T} J'g_s\zeta_{st} \right\|^2 \\
& + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}(\tilde{g}_s - J'g_s)\eta_{st} \right\|^2 + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T} J'g_s\eta_{st} \right\|^2 \\
& + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T}(\tilde{g}_s - J'g_s)\xi_{st} \right\|^2 \\
& + \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0} \left\| \frac{1}{T}\sum\nolimits_{s=1}^{T} J'g_s\xi_{st} \right\|^2) \left\|V_{NT}^{-1}\right\|^2 \\
=\; & 8(IX + X + XI + XII + XIII + XIV + XV + XVI) \left\|V_{NT}^{-1}\right\|^2 . \qquad (37)
\end{aligned}
$$

Consider each term one by one.

$$
\begin{aligned}
IX &\leq (\frac{1}{T}\sum\nolimits_{s=1}^{T}\left\|\tilde{g}_s - J'g_s\right\|^2) \sup_{k\in D, k<k_0} \frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}\frac{1}{T}\sum\nolimits_{s=1}^{T}|\gamma_N(s,t)|^2 \\
&= O_p(\frac{1}{\delta_{NT}^2})O_p(\frac{1}{T}).
\end{aligned}
$$

$$X \leq \|J\|^2 (\frac{1}{T}\sum_{s=1}^{T} \|g_s\|^2) \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \frac{1}{T}\sum_{s=1}^{T} |\gamma_N(s,t)|^2$$
$$= O_p(1)O_p(1)O_p(\frac{1}{T}),$$

where the equalities are explained in proof of term I.

$$XI \leq (\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{g}_s - J'g_s\right\|^2)\frac{1}{N}(\frac{1}{T}\sum_{s=1}^{T} \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2)$$
$$= O_p(\frac{1}{\delta_{NT}^2})\frac{1}{N}O_p(1).$$

$$XII \leq \|J\|^2 (\frac{1}{T}\sum_{s=1}^{T} \|g_s\|^2)\frac{1}{N}(\frac{1}{T}\sum_{s=1}^{T} \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2)$$
$$= O_p(1)O_p(1)\frac{1}{N}O_p(1),$$

where the equalities follow from part (1) of Assumption 10.

$$XIII \leq (\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{g}_s - J'g_s\right\|^2) \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \frac{1}{T}\sum_{s=1}^{T} \left|\frac{1}{N}\sum_{i=1}^{N} g_s'\gamma_i e_{it}\right|^2$$
$$\leq (\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{g}_s - J'g_s\right\|^2)(\frac{1}{T}\sum_{s=1}^{T} \|g_s\|^2)\frac{1}{N} \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \gamma_i e_{it}\right\|^2$$
$$= O_p(\frac{1}{\delta_{NT}^2})O_p(1)\frac{1}{N}O_p(1).$$

$$XIV \leq \|J\|^2 \left\|\frac{1}{T}\sum_{s=1}^{T} g_s g_s'\right\|^2 \frac{1}{N} \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \gamma_i e_{it}\right\|^2$$
$$= O_p(1)O_p(1)\frac{1}{N}O_p(1),$$

where the equalities follow from part (2) of Assumption 10.

$$XV \leq (\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{g}_s - J'g_s\right\|^2) \sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \frac{1}{T}\sum_{s=1}^{T} \left\|\frac{1}{N}\sum_{i=1}^{N} g_t'\gamma_i e_{is}\right\|^2$$
$$\leq (\frac{1}{T}\sum_{s=1}^{T} \left\|\tilde{g}_s - J'g_s\right\|^2)(\sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \|g_t\|^2)\frac{1}{N}(\frac{1}{T}\sum_{s=1}^{T} \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \gamma_i e_{is}\right\|^2)$$
$$= O_p(\frac{1}{\delta_{NT}^2})O_p(1)\frac{1}{N}O_p(1).$$

$$XVI \leq \|J\|^2 (\frac{1}{T}\sum_{s=1}^{T} \|g_s\|^2)(\sup_{k\in D, k<k_0} \frac{1}{k_0-k} \sum_{t=k+1}^{k_0} \|g_t\|^2)\frac{1}{N}(\frac{1}{T}\sum_{s=1}^{T} \left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N} \gamma_i e_{is}\right\|^2)$$
$$= O_p(1)O_p(1)O_p(1)\frac{1}{N}O_p(1),$$

where the equalities follow from $E(\frac{1}{T}\sum_{s=1}^{T}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{is}\right\|^2) \leq M$, which follows from part (ii) of Lemma 1 in Bai and Ng (2002).

Finally consider part (7).

$$\begin{aligned}\sup_{k\leq k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}(\tilde{g}_t - J'g_t)g_t'J\right\| &\leq \sup_{k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}(\tilde{g}_t - J'g_t)g_t'J\right\| \\ &+ \left\|\frac{1}{T-k_0}\sum\nolimits_{t=k_0+1}^{T}(\tilde{g}_t - J'g_t)g_t'J\right\|.\end{aligned}$$

Based on parts (3) and (4), the first term is $O_p(\frac{1}{\delta_{NT}})$. Following the same procedure as part (2), it can be shown the second term is also $O_p(\frac{1}{\delta_{NT}})$. ■

Lemma 6 *Under Assumptions 1-9, terms (1)-(7) in Lemma 5 are* $o_p(1)$.

Proof. The results can be proved following the same procedure as proving Lemma 5, the differences are stated below. Assumption 10 is used in the proof of III, IV, XI, XII, V, VI, $XIII$, XIV to calculate the stochastic order of $\sup\limits_{k\in D^c,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2$, $\sup\limits_{k\in D,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2$, $\sup\limits_{k\in D^c,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\gamma_i e_{it}\right\|^2$ and $\sup\limits_{k\in D,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\right.$ Without Assumption 10, all are no longer necessarily $O_p(1)$. Nevertheless, we can use Lemma 4 to show that all are $O_p(\log T)$ without making any dependence Assumption on the error process.

Denote $X_t = \left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2$, then $\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2 = \frac{1}{k}\sum_{t=1}^{k} X_t$. Taking $r = 1$, $\beta_k = k$ and $\alpha_l = M$, then for each m with $1 \leq m \leq T$,

$$E(\sup_{1\leq k\leq m}|S_k|) = E(S_m) \leq mM \leq \sum\nolimits_{k=1}^{m}\alpha_k, \tag{38}$$

hence by Lemma 4,

$$E(\sup_{1\leq k\leq k_0}\left|\frac{S_k}{k}\right|) \leq 4\sum\nolimits_{k=1}^{k_0}\frac{M}{k} \leq 4M\log T + 4M\gamma, \tag{39}$$

where γ is the Euler-Mascheroni constant. It follows that $\sup\limits_{k\in D^c,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2$ and $\sup\limits_{k\in D,k\leq k_0}\frac{1}{k}\sum_{t=1}^{k}\left|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\right|^2$ are both $O_p(\log T)$. All other terms can be proved to be $O_p(\log T)$ similarly. Now $III = O_p(\frac{\sqrt{\log T}}{\sqrt{N}\delta_{NT}})$, $IV = O_p(\sqrt{\frac{\log T}{N}})$, $V = O_p(\frac{\sqrt{\log T}}{\sqrt{N}\delta_{NT}})$, $VI = O_p(\sqrt{\frac{\log T}{N}})$, $XI = O_p(\frac{\log T}{N\delta_{NT}^2})$, $XII = O_p(\frac{\log T}{N})$, $XIII = O_p(\frac{\log T}{N\delta_{NT}^2})$ and $XIV = O_p(\frac{\log T}{N})$. With Assumption 9, all terms are $o_p(1)$. ■

Lemma 7 *Under Assumptions 1-8 and 9 or 10,* *(1)* $\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}z_t\right\|^2=o_p(1)$, *(2)* $\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}z_t\right\|^2=o_p(1)$,

(3) $\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|=o_p(1)$, *(4)* $\sup_{k\in D,k\le k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|=o_p(1)$,

(5) $\sup_{k\in D^c,k<k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}z_t\right\|=o_p(1)$, *(6)* $\sup_{k\in D,k<k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}z_t\right\|=o_p(1)$,

(7) $\sup_{k\in D^c,k<k_0}\frac{1}{k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}z_t\right\|^2=o_p(1)$, *(8)* $\sup_{k\in D,k<k_0}\frac{1}{k_0}\frac{1}{k_0-k}\left\|\sum_{t=k+1}^{k_0}z_t\right\|^2=o_p(1)$,

(9) $\sup_{k\le k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T}z_t\right\|=o_p(1)$.

Proof. We will prove the results under Assumptions 1-8 and 10 first. Under Assumptions 1-9, the proof follows the same procedure, except for using Lemma 6 instead of Lemma 5. Recall that $z_t=vec[(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)']+vec[(\tilde{g}_t-J'g_t)g_t'J]+vec[J'g_t(\tilde{g}_t-J'g_t)']$
$+vec[(J'-J_0')g_tg_t'(J-J_0)]+vec[(J'-J_0')g_tg_t'J_0]+vec[J_0'g_tg_t'(J-J_0)]$.

For parts (1) and (2),

$$\begin{aligned}\left\|\sum_{t=1}^{k}z_t\right\|^2 &\le (\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)'\right\|+2\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\| \\ &\quad+\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|+2\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|)^2 \\ &\le 4\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)'\right\|^2+16\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\|^2 \\ &\quad+4\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|^2+16\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|^2. \qquad (40)\end{aligned}$$

Consider the four terms one by one.

Using Lemma 1,

$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)'\right\|^2\le\frac{1}{\tau_0(\tau_0-\eta)}(\frac{1}{T}\sum_{t=1}^{T}\left\|\tilde{g}_t-J'g_t\right\|^2)^2=O_p(\frac{1}{\delta_{NT}^4}).$$

Using part (6) of Lemma 5,

$$\begin{aligned}\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)'\right\|^2 &\le \sup_{k\in D^c,k\le k_0}\left\|\frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)(\tilde{g}_t-J'g_t)'\right\|^2 \\ &= O_p(\frac{1}{\delta_{NT}^4}).\end{aligned}$$

Using part (1) of Lemma 5,

$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\|^2\le\sup_{k\in D,k\le k_0}\left\|\frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\|^2=O_p(\frac{1}{\delta_{NT}^2}).$$

Using part (2) of Lemma 5,

$$\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\|^2\le\sup_{k\in D^c,k\le k_0}\left\|\frac{1}{k}\sum_{t=1}^{k}(\tilde{g}_t-J'g_t)g_t'J\right\|^2=O_p(\frac{1}{\delta_{NT}^2}).$$

Using Lemma 2 and Assumption 3,

$$\sup_{k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|^2\le\|J-J_0\|^4\sup_{k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|^2=o_p(1),$$

$$\sup_{k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|^2\le\|J-J_0\|^2\|J_0\|^2\sup_{k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|^2=o_p(1).$$

Hence $\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|^2$, $\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|^2$, $\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|^2$ and $\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|^2$ are all $o_p(1)$. It follows $\sup_{k\in D,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}z_t\right\|^2$ and $\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\frac{1}{k}\left\|\sum_{t=1}^{k}z_t\right\|^2$ are both $o_p(1)$.

For parts (3) and (4),

$$\begin{aligned}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}z_t\right\| \le\ & \frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)(\tilde g_t-J'g_t)'\right\|+2\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)g_t'J\right\|\\ &+\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|+2\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|. \quad (41)\end{aligned}$$

Using Lemma 1,

$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)(\tilde g_t-J'g_t)'\right\|\le\frac{1}{\tau_0}\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\|\tilde g_t-J'g_t\right\|^2=O_p(\frac{1}{\delta_{NT}^2}),$$

$$\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)(\tilde g_t-J'g_t)'\right\|\le\frac{1}{\tau_0}\frac{1}{T}\sum\nolimits_{t=1}^{T}\left\|\tilde g_t-J'g_t\right\|^2=O_p(\frac{1}{\delta_{NT}^2}).$$

Using part (1) of Lemma 5,

$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)g_t'J\right\|\le\sup_{k\in D,k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)g_t'J\right\|=O_p(\frac{1}{\delta_{NT}}).$$

Using part (2) of Lemma 5,

$$\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)g_t'J\right\|\le\sup_{k\in D^c,k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}(\tilde g_t-J'g_t)g_t'J\right\|=O_p(\frac{1}{\delta_{NT}}).$$

Using Lemma 2 and Assumption 3,

$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|\le\|J-J_0\|^2\sup_{k\in D,k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|=o_p(1),$$

$$\sup_{k\in D^c,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'(J-J_0)\right\|\le\|J-J_0\|^2\sup_{k\in D^c,k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|=o_p(1),$$

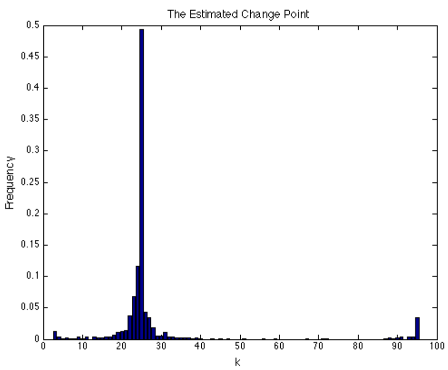
$$\sup_{k\in D,k\le k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|\le\|J-J_0\|\,\|J_0\|\sup_{k\in D,k\le k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|=o_p(1),$$

$$\sup_{k\in D^c,k\leq k_0}\frac{1}{k_0}\left\|\sum\nolimits_{t=1}^{k}(J'-J_0')g_tg_t'J_0\right\|\leq\|J-J_0\|\,\|J_0\|\sup_{k\in D^c,k\leq k_0}\left\|\frac{1}{k}\sum\nolimits_{t=1}^{k}g_tg_t'\right\|=o_p(1).$$

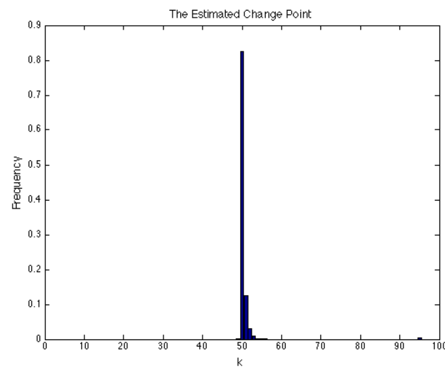
It follows that $\sup_{k\in D,k\leq k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|$ and $\sup_{k\in D^c,k\leq k_0}\frac{1}{k_0}\left\|\sum_{t=1}^{k}z_t\right\|$ are both $o_p(1)$. parts (5), (6), (7), (8) and (9) can be proved following the same procedure. More specifically, part (5) uses Lemma 1, Lemma 2, part (3) of Lemma 5 and Lemma 3; part (6) uses parts (5) and (4) of Lemma 5, Lemma 2 and Lemma 3; parts (7) and (8) follow from (5) and (6) respectively; part (9) uses Lemma 1, Lemma 2, part (7) of Lemma 5 and $\sup_{k\leq k_0}\left\|\frac{1}{T-k}\sum_{t=k+1}^{T}g_tg_t'\right\|=O_p(1)$, which is proved below.

$$\begin{aligned}\sup_{k\leq k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}g_tg_t'\right\| &\leq \sup_{k<k_0}\left\|\frac{1}{T-k}\sum\nolimits_{t=k+1}^{T}g_tg_t'\right\|+\left\|\frac{1}{T-k_0}\sum\nolimits_{t=k_0+1}^{T}g_tg_t'\right\|\\ &\leq \sup_{k<k_0}\left\|\frac{1}{k_0-k}\sum\nolimits_{t=k+1}^{k_0}g_tg_t'\right\|+2\left\|\frac{1}{T-k_0}\sum\nolimits_{t=k_0+1}^{T}g_tg_t'\right\|=O_p(1).\end{aligned}$$

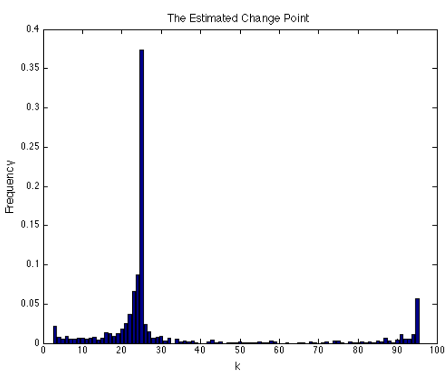
■



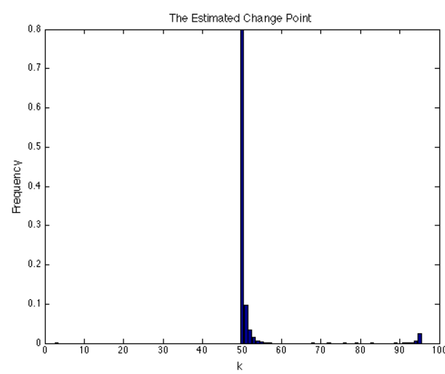
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.68$, $sd(\tilde{r}) = 0.60$



$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 6.85$, $sd(\tilde{r}) = 0.38$

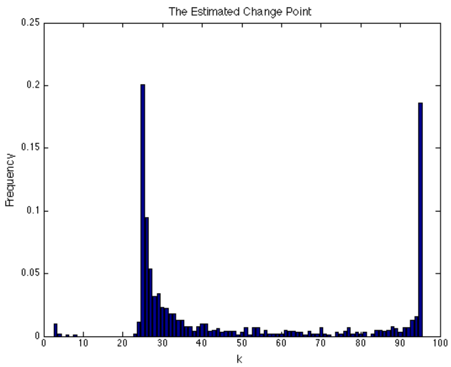


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.75$, $sd(\tilde{r}) = 0.58$

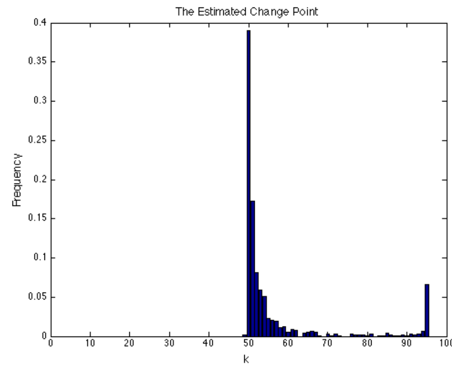


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 6.74$, $sd(\tilde{r}) = 0.48$

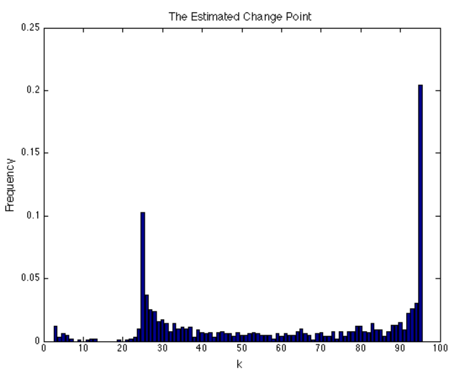
Figure 1: Histogram of $\tilde{k}$ for $(N, T) = (100, 100), (r_1, r_2, r + q_1) = (3, 5, 7)$



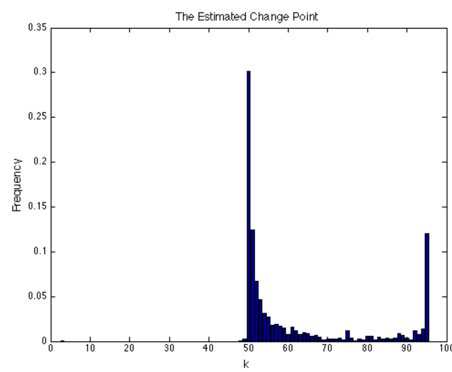
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$



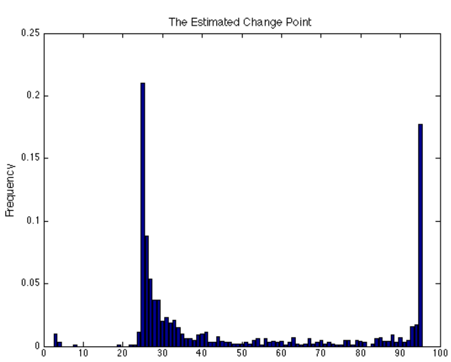
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 4.98$, $sd(\tilde{r}) = 0.13$



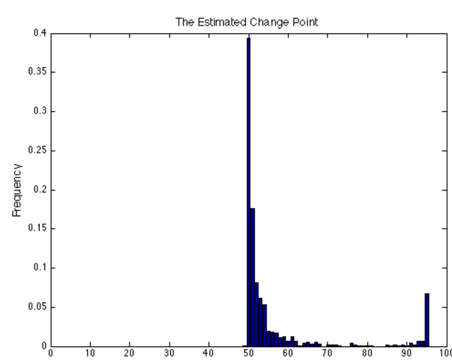
$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$



$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0.06$

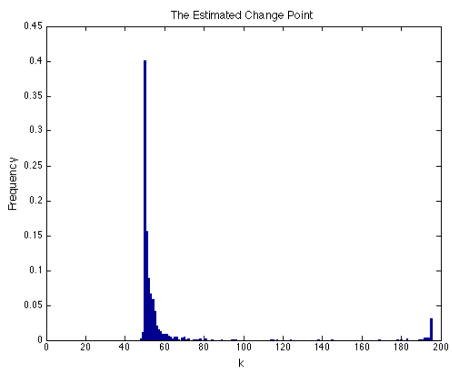


$(\rho, \alpha, \beta) = (0, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$

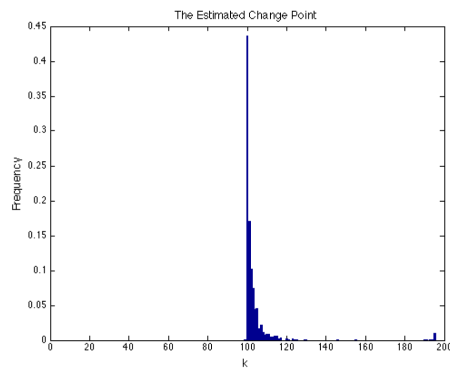


$(\rho, \alpha, \beta) = (0, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 4.99$, $sd(\tilde{r}) = 0.08$

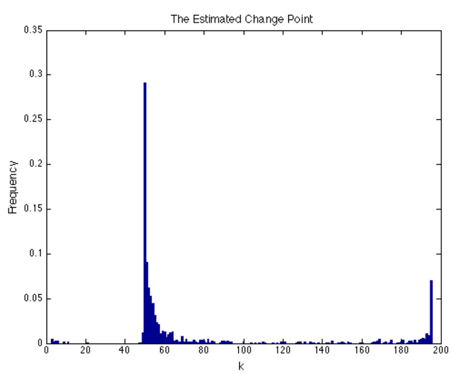
Figure 2: Histogram of $\tilde{k}$ for $(N, T) = (100, 100), (r_1, r_2, r + q_1) = (3, 5, 5)$



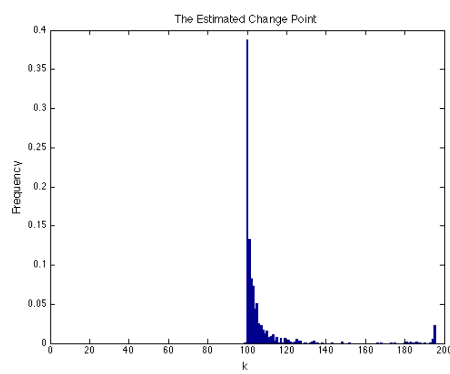
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$



$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$

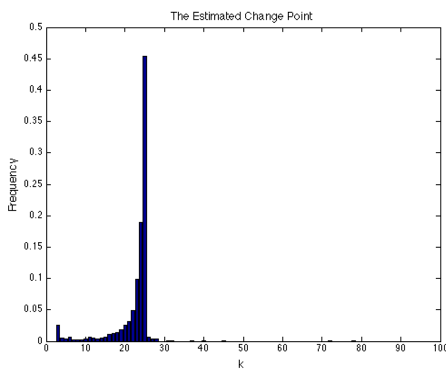


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$

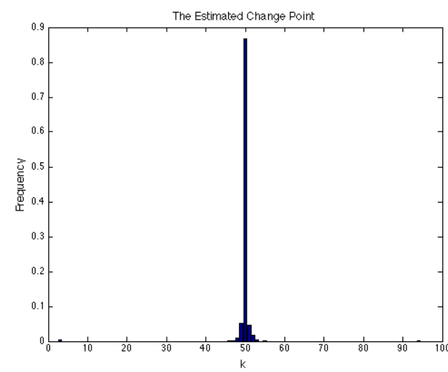


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0.04$

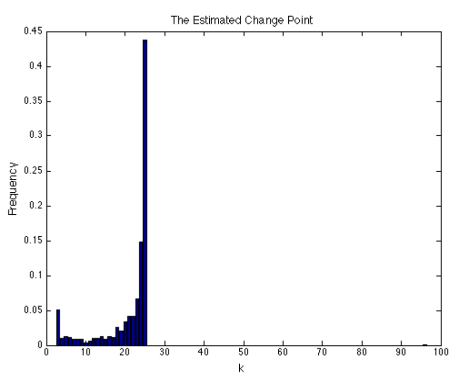
Figure 3: Histogram of $\tilde{k}$ for $(N, T) = (100, 200), (r_1, r_2, r + q_1) = (3, 5, 5)$



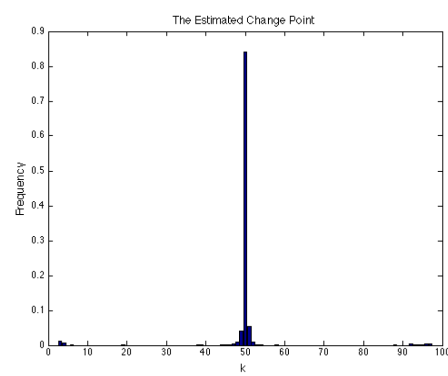
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 4.51$, $sd(\tilde{r}) = 0.56$



$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$

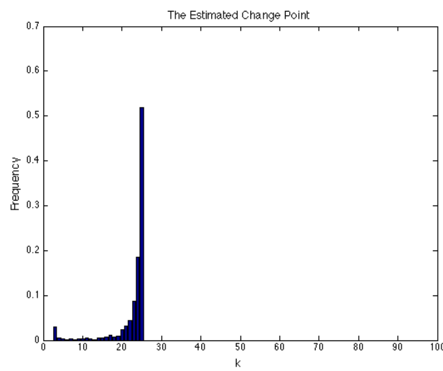


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 4.86$, $sd(\tilde{r}) = 0.35$

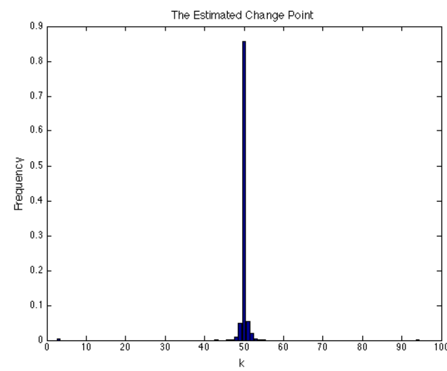


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0$

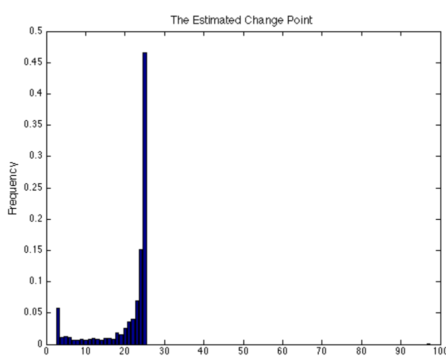
Figure 4: Histogram of $\tilde{k}$ for $(N, T) = (100, 100), (r_1, r_2, r + q_1) = (3, 3, 5), a = 1$



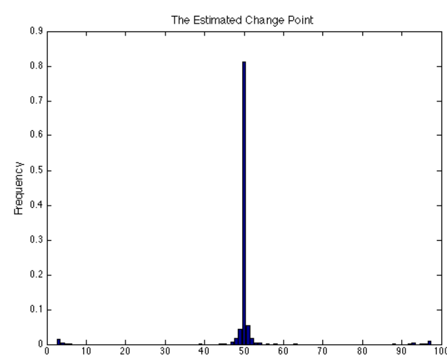
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0.08$



$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.00$, $sd(\tilde{r}) = 0.04$

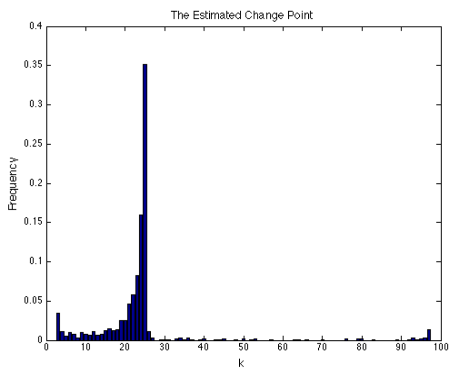


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 6.03$, $sd(\tilde{r}) = 1.16$

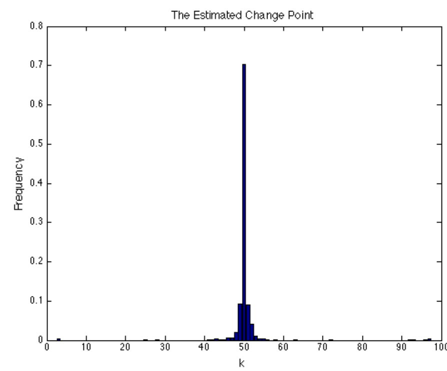


$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 6.06$, $sd(\tilde{r}) = 1.12$

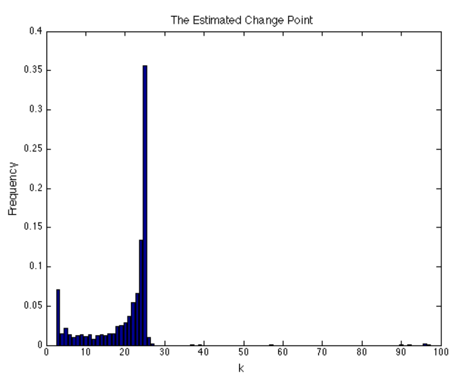
Figure 5: Histogram of $\tilde{k}$ for $(N, T) = (100, 100), (r_1, r_2, r + q_1) = (3, 3, 5), a = 0.6$



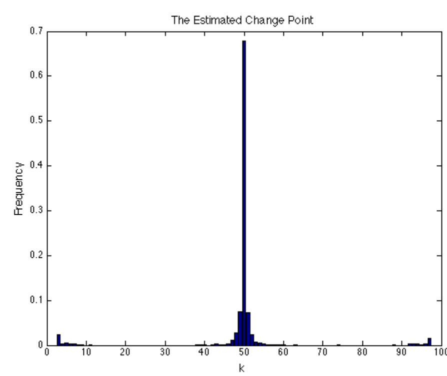
$(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 4.27$, $sd(\tilde{r}) = 0.60$



$\tau_0 = 0.5$, $(\rho, \alpha, \beta) = (0, 0, 0)$, homogeneous
R^2, $ave(\tilde{r}) = 4.85$, $sd(\tilde{r}) = 0.36$



$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.25$, $ave(\tilde{r}) = 5.60$, $sd(\tilde{r}) = 1.17$



$(\rho, \alpha, \beta) = (0.5, 0.2, 0.2)$, heterogeneous R^2,
$\tau_0 = 0.5$, $ave(\tilde{r}) = 5.94$, $sd(\tilde{r}) = 1.08$

Figure 6: Histogram of $\tilde{k}$ for $(N, T) = (100, 100), (r_1, r_2, r + q_1) = (3, 3, 5), a = 0.2$

Table 1: Estimated number of pre-break and post-break factors and estimated factor space for $r_1 = 3, r_2 = 5, r + q_1 = 7$

N	T	$\tau_0 = 0.25$					$\tau_0 = 0.5$				
		IC_{p2}		GR			IC_{p2}		GR		
		$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$
$\rho = 0, \alpha = 0, \beta = 0$, homogeneous R^2											
100	100	4/8	2/2	11/7	5/1	0.94	0/0	13/0	0/1	2/0	0.96
100	200	0/0	0/0	0/0	0/0	0.95	0/0	0/0	0/0	0/0	0.96
200	200	0/0	0/0	0/0	0/0	0.98	0/0	0/0	0/0	0/0	0.98
200	400	0/0	0/0	0/0	0/0	0.98	0/0	0/0	0/0	0/0	0.98
$\rho = 0.5, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2											
100	100	3/13	2/3	23/4	5/2	0.95	0/4	8/1	1/2	10/0	0.97
100	200	0/2	0/0	2/0	0/1	0.96	0/0	0/0	0/0	0/0	0.97
200	200	0/1	0/3	2/0	0/1	0.98	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/0	0/0	0/0	0.98	0/0	0/0	0/0	0/0	0.99

Table 2: Estimated number of pre-break and post-break factors and estimated factor space for $r_1 = 3, r_2 = 5, r + q_1 = 5$

N	T	$\tau_0 = 0.25$					$\tau_0 = 0.5$				
		IC_{p2}		GR			IC_{p2}		GR		
		$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$
$\rho = 0, \alpha = 0, \beta = 0$, homogeneous R^2											
100	100	3/41	15/6	9/39	29/0	0.91	0/10	18/2	0/9	12/0	0.96
100	200	0/6	2/1	0/6	5/0	0.95	0/2	1/0	0/1	1/0	0.96
200	200	0/6	2/0	0/5	4/0	0.97	0/1	0/0	0/1	0/0	0.98
200	400	0/1	1/0	0/1	1/0	0.98	0/0	0/0	0/0	0/0	0.98
$\rho = 0.5, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2											
100	100	1/68	20/14	10/59	46/0	0.89	0/26	13/6	1/20	30/0	0.96
100	200	0/27	5/4	2/22	13/0	0.94	0/6	1/2	0/5	4/0	0.97
200	200	0/31	4/5	1/24	14/0	0.95	0/7	1/1	0/6	5/0	0.98
200	400	0/7	1/1	0/5	4/0	0.98	0/2	0/0	0/1	1/0	0.99
$\rho = 0, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2											
100	100	1/43	11/7	9/38	28/0	0.91	0/11	9/2	0/9	12/0	0.96
100	200	0/6	1/1	0/6	4/0	0.96	0/2	0/0	0/1	1/0	0.97
200	200	0/9	1/0	0/5	4/0	0.98	0/1	0/0	0/0	0/0	0.98
200	400	0/1	0/0	0/1	1/0	0.98	0/0	0/0	0/0	0/0	0.98

Table 3: Estimated number of pre-break and post-break factors and estimated factor space for $r_1 = 3, r_2 = 3, r + q_1 = 5$

N	T	$\tau_0 = 0.25$					$\tau_0 = 0.5$				
		IC_{p2}		GR			IC_{p2}		GR		
		$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$	$\tilde{r}_1$	$\tilde{r}_2$	$\tilde{r}_1$	$\tilde{r}_2$	$R^2_{\tilde{F},F}$
$\rho = 0, \alpha = 0, \beta = 0$, homogeneous R^2, $a = 1$											
100	100	5/4	0/1	14/0	0/1	0.97	0/0	0/0	0/0	0/0	0.97
100	200	0/0	0/0	1/0	0/0	0.97	0/0	0/0	0/0	0/0	0.97
200	200	0/0	0/0	0/0	0/0	0.98	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/0	0/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
$\rho = 0.5, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2, $a = 1$											
100	100	3/9	0/8	27/0	0/4	0.97	1/4	0/4	2/1	1/2	0.97
100	200	0/2	0/4	4/0	0/2	0.98	0/1	0/0	0/0	0/0	0.98
200	200	0/1	0/3	2/0	0/2	0.99	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/1	1/0	0/1	0.99	0/0	0/0	0/0	0/0	0.99
$\rho = 0, \alpha = 0, \beta = 0$, homogeneous R^2, $a = 0.6$											
100	100	4/3	0/1	12/0	0/0	0.97	0/0	0/0	0/0	0/0	0.97
100	200	0/0	0/0	1/0	0/0	0.97	0/0	0/0	0/0	0/0	0.97
200	200	0/0	0/0	0/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/0	0/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
$\rho = 0.5, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2, $a = 0.6$											
100	100	3/9	0/6	26/0	0/3	0.98	1/2	0/3	2/2	2/2	0.98
100	200	0/2	0/3	3/0	0/1	0.98	0/1	0/1	0/0	0/0	0.98
200	200	0/1	0/3	2/0	0/1	0.99	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/1	1/0	0/1	0.99	0/0	0/0	0/0	0/0	0.99
$\rho = 0, \alpha = 0, \beta = 0$, homogeneous R^2, $a = 0.2$											
100	100	5/8	0/1	18/0	2/0	0.97	0/0	0/0	0/0	1/0	0.97
100	200	2/5	3/7	10/0	16/0	0.97	0/1	1/0	2/0	1/0	0.97
200	200	0/0	0/0	1/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/0	0/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
$\rho = 0.5, \alpha = 0.2, \beta = 0.2$, heterogeneous R^2, $a = 0.2$											
100	100	5/13	0/0	33/0	0/0	0.98	1/2	1/2	3/0	2/0	0.98
100	200	1/3	0/0	7/0	4/0	0.98	0/0	0/0	0/0	1/0	0.98
200	200	0/2	0/0	3/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99
200	400	0/0	0/0	1/0	0/0	0.99	0/0	0/0	0/0	0/0	0.99